The Globalization Problem for inner automorphisms and Skolem-Noether Theorems

Jeremy Haefner
Department of Mathematics
University of Colorado
Colorado Springs CO
60933 U.S.A.
haefner@math.uccs.edu

Ángel del Río*
Departamento de Matemáticas
Universidad de Murcia
30071. Murcia. Spain
adelrio@fcu.um.es

Abstract

We investigate the circumstances under which an inner automorphism of a ring with local units can be built from "local" information. Specifically, we consider three natural "inner" type properties for an automorphism of a ring with local units. We show that every inner automorphism is locally inner but the converse is false, even if the automorphism is "piecewise" inner. On the other hand, we construct a large class of rings for which every locally inner automorphism is actually inner. Finally we obtain some Skolem-Noether type Theorems for infinite matrix and triangular matrix rings.

1 Introduction and preliminaries

We recall (see e.g. [2]) that a ring A is said to have *local units* provided there exists a set of idempotents E of A such that, for every finite subset $\mathcal{F}$ of A, there is an idempotent $e \in E$ such that $\mathcal{F} \subset eAe$. The set E is said to be a set of local units for A. A natural technique in the study of rings with local units is to try to obtain global information of A from the local unital rings eAe $(e \in E)$ of A. In this paper, we investigate the relationship between an inner automorphism and its behavior on the local unital rings eAe $(e \in E)$.

Inner automorphisms for rings with local units were introduced in [4] but the following definition comes from [4] and [1]:

Definition and Theorem 1.1 ([4], [1]) Let A be a ring with local units and let $\alpha \in \mathrm{Aut}(A)$. We order the set of idempotents of A via $e \leq f$ if and only if $eAe \subset fAf$. We say that α is **inner** if any of the following equivalent conditions holds:

---

*Research partially supported by DGES and Comunidad Autónoma de Murcia

1. for all $e = e^2 \in A$, (alternatively, for every $e \in E$, being E a set of local units) there exist $u(e) \in \alpha(e)Ae$, and $v(e) \in eA\alpha(e)$, such that for $x \in eAe$, $\alpha(x) = u(e)xv(e)$ and any one of the following equivalent compatibility conditions holds for every two idempotents e, f (for every $e, f \in E$) such that $e \leq f$:
    (a) $u(f)e = u(e)$.
    (b) $ev(f) = v(e)$.
2. α is in the kernel of the natural group homomorphism $\mathrm{Aut}(A) \to \mathrm{Pic}(A)$.
3. α is the restriction of an inner automorphism of $Q(A)$, the ring of multipliers of A in $\mathrm{End}(A_A)$.

Having defined inner automorphisms globally, we turn our attention to "local" definitions of inner automorphisms. In particular, by considering the behavior of an automorphism when it is restricted to eAe, we can define automorphisms that are "piecewise inner" and "locally inner".

Definition 1.2 Let $\alpha \in \mathrm{Aut}A$. For every idempotent e of A, let $\alpha_e : eAe \to \alpha(e)A\alpha(e)$ denote the restriction of α to an isomorphism between eAe and $\alpha(e)A\alpha(e)$.

- We say that an idempotent e is **α-invariant** provided $\alpha(e) = e$ and we say that a set of local units E of A is **α-invariant** provided each $e \in E$ is α-invariant.
- The automorphism $\alpha \in \mathrm{Aut}A$ is **piecewise constructed** (or **piecewise defined**) if A has a set of α-invariant local units E.
- The automorphism α is **piecewise inner** if A has a set of α-invariant local units E so that for every $e \in E$, α_e is an inner automorphism (in the classical sense) of eAe.
- If e is an idempotent of A, we say that α is **inner at** e provided there are elements $u, v \in A$ such that for all $x \in eAe$, $\alpha(x) = uxv$. Note that if $\alpha(e) = e$, where $e \in E$, then α_e is inner (in the classical sense) if and only if α is inner at e.
- We say that α is **locally inner** provided α is inner at each idempotent e of A.

The main focus of this paper is to determine if every locally inner (piecewise inner) automorphism is inner. While we find that there are locally inner automorphisms that are NOT inner, we also find a large class of rings with local units for which all locally inner automorphisms (and hence all piecewise inner automorphisms) are inner. See Theorem 2.4.

The global/local results of inner automorphisms discussed above also have relevance for unital rings. Assume that A is the ring of multipliers (also known as the translation hull) for a ring I with local units ; see [1] or [13] for more details. The idea is to obtain properties of A from properties of the unital rings eIe, where e ranges over a set of local units of I. In this paper, we use this technique to extend the Skolem-Noether result to certain infinitely generated (unital) algebras. See Theorems 4.1, 4.3, 4.4, and 4.5.

We close this section with some comments about some immediate relations between inner, locally inner and piecewise inner automorphisms.

Remark 1.3 Let α denote an automorphism of a ring with local units A.

1. By definition, every inner automorphism is locally inner.
2. If the automorphism α is inner at e and $f \leq e$, then α is inner at f. This shows that α is locally inner if α is inner at all the elements of a set of local units.
3. An automorphism α is piecewise inner if and only if it is locally inner and piecewise defined. To see this, assume that α is inner at e and let $u, v \in A$, such that $\alpha(x) = uxv$ for every $x \in eAe$. By replacing u and v by $\alpha(e)ue$ and $ev\alpha(e)$, respectively, we may assume that $u \in \alpha(e)Ae$ and $eA\alpha(e)$. If, in addition, e is α-invariant, then $uv = uev = \alpha(e) = e$ and $vu = \alpha^{-1}(\alpha(vu)) = \alpha^{-1}(uvuv) = \alpha^{-1}(e) = e$, and so α_e is inner. This shows that if α is locally inner and piecewise defined, then it is piecewise inner. The converse is straightforward.
4. If A is unital then all definitions (inner, locally inner, and piecewise inner) coincide with the classical one; this follows from the previous arguments.

2 Rings for which every locally inner automorphism is inner.

In this section, we give a large class of rings with local units for which every locally inner automorphism is inner. Our method relies on transfinite induction on well founded sets (see e.g. [12]). An ordered set is said to be *well founded* if every nonempty subset contains a minimal element. We recall the formal set theoretical statement of the Recursion Theorem for Well Founded Sets ([12, Theorem III.5.6]): Let $\mathcal{V}$ denote the class of all the ZFC objects, $(A, \leq)$ a well founded set, and $\mathcal{L} : A \times \mathcal{V} \to \mathcal{V}$ a class map. Then there exists a unique map $l : A \to \mathcal{V}$ such that $l(a) = \mathcal{L}(a, \{(x, l(x)) \mid x < a\})$ for every $a \in A$. Well founded sets appear in Set Theory in connection with the Axiom of Foundation which is useful in proofs of relative consistence results (see Chapters III and IV of [12]). We are not aware of any use in Ring Theory of induction of recursion techniques in well founded set. The existence of well founded sublattices in Boolean algebras has been investigated in [5] and [6].

Lemma 2.1 *Every set of local units contains a well founded set of local units.*

Proof. Let E be a set of local units for the ring A with the usual ordering. By the Axiom of Choice, there is a map $f : \mathcal{F} \to E$, where $\mathcal{F}$ is the set of finite subsets of A and $F \subseteq f(F)Af(F)$ for every $F \in \mathcal{F}$. Since $\mathcal{F}$ is well founded by inclusion, there is a unique map $e : \mathcal{F} \to A$ so that $e(F) = f(F \cup \{e(F_1) \mid F_1 \subset F\})$ for every $F \in \mathcal{F}$ by the Recursion Theorem for Well Founded Sets. (Here the class map $\mathcal{L} : \mathcal{F} \times \mathcal{V} \to \mathcal{V}$ which induces e is given by $\mathcal{L}(F, X) = f(F \cup \text{im } X)$, if X is a map whose image is a finite subset of A and $\emptyset$ otherwise.) Since e preserves the order, the image of e is a well founded set of local units. Note that "$\subset$" means properly contained. ▮

Let e and f be two idempotents of A. Then $Hom_A(Ae, Af) \simeq eAf$ and we consider this isomorphism as an identification. In particular eAe and fAf are identified with the endomorphism ring of ${}_AAe$ and ${}_AAf$ respectively. Then any module isomorphism ${}_AAe \simeq {}_AAf$ induces a ring isomorphism $eAe \simeq fAf$. Moreover, the isomorphisms $Ae \simeq Af$ correspond, by the previous identification, to elements $v \in eAf$ such that there exists $u \in fAe$ so that $uv = f$ and $vu = e$. More concretely, for such u and v, the map $\phi : Ae \to Af$ given by $\phi(x) = xv$ is an isomorphism of left A-modules with inverse $\phi^{-1}(x) = xu$. In this case, the isomorphism $\alpha : eAe \to fAf$ induced by ϕ_u is given by $\alpha(x) = uxv$. It follows that an automorphism α of A is inner at e if and only if the isomorphism $\alpha_e : eAe \simeq \alpha(e)A\alpha(e)$ is induced by a module isomorphism $\phi_e : Ae \to A\alpha(e)$.

Suppose now that α is locally inner so that for every idempotent e of A, α_e is induced by an isomorphism $\phi_e : Ae \to A\alpha(e)$. If E is a set of local units of A, then we have that $A = \lim_{e\in E} Ae = \lim_{e\in E} A\alpha(e)$. If it happens that $\Phi = (\phi_e : Ae \to A\alpha(e) : e \in E)$ is a direct system of module isomorphisms, then Φ induces an isomorphism $\phi : {}_AA \to {}_AA$ which, in turn, induces α. Then considering A as a subring of its multiplier ring $Q(A)$ which itself is considered as a subring of $\mathrm{End}({}_AA)$, we have that α is inner. (See [1].)

In general, however, Φ is not a direct system. In fact this only happens when the compatibility conditions (either (1) or (2)) of Definition 1.2 hold. But the system Φ that induces α_e at every e is not unique. Our method consists of modifying Φ so that it becomes a direct system. In order to do this, we need to understand how we can modify Φ without changing the automorphism associated to it. This is the task of next lemma.

Lemma 2.2 *Let e and f be two idempotents of A and let $u, u' \in fAe$ and $v, v' \in eAf$ so that $uv = u'v' = f$ and $vu = v'u' = e$. Then the following are equivalent:*

1. *For every $x \in eAe$, $uxv = u'xv'$.*
2. *vu' is a central unit of eAe with inverse $v'u$.*
3. *There is a central unit c of eAe so that $u' = uc$ and $v' = c^{-1}v$.*

Therefore two module isomorphisms $\phi, \psi : Ae \to Af$ induce the same ring isomorphism $eAe \to fAf$ if and only if there is a central unit c of eAe so that $\psi(x) = \phi(xc)$ for every $x \in Ae$.

Proof. 1 implies 2. Assume that $uxv = u'xv'$ for every $x \in eAe$. Then $xvu' = vuxvu' = vu'xv'u' = vu'x$. Moreover $vu'v'u = vfu = vu = e$ and similarly $v'uvu' = e$. Therefore vu' is a central unit of eAe.

2 implies 3. Notice that $u' = uvu'$ and $v' = v'uv$, so that $c = vu'$ suffices.

3 implies 1. Let c be a central unit of eAe so that $u' = uc$ and $v' = c^{-1}v$. Then $u'xv' = ucxc^{-1}v = uxv$. The final statement follows from the observation made above. ▮

Before giving the main result of this section we adopt the following notation:

Notation 2.3 For every $e = e^2 \in A$, $Z(e)$ denotes the center of eAe and $Z(e)^*$ the group of units of $Z(e)$. For every set E of idempotents of A, let

$$Z(E) = \{(x_e)_{e\in E} \in \prod_{e\in E} Z(e) \mid x_e f = x_f \text{ for every } f \leq e\}$$

and $Z(E)^* = Z(E) \cap \prod_{e\in E} Z(e)^*$. If E is a set of idempotents of A and $e \in E$, then set $E_e = \{f \in E : f < e\}$ and let $\lambda_e : Z(e) \to Z(E_e)$ be the map given by $\lambda_e(x) = (xf)_{f\in E_e}$. Note that if e is a minimal element of E, then $E_e = \emptyset$; in this case we consider $Z(E_e)$ as a singleton set. The map λ_e restricts to a map $Z(e)^* \to Z(E_e)^*$ that we denote also by λ_e.

For every $f \in E$, let $\pi_f : \prod_{e\in E} Z(e) \to Z(f)$ be the projection map. Note that if $f \in E_e$, then $xf = \pi_f \lambda_e(x)$ for every $x \in Z(e)$.

We now present the crux of our argument.

Theorem 2.4 *Assume A is a ring with local units such that there is a collection of local units E for which*

1. *$\leq$ is well founded in E, and*
2. *for every non minimal element $e \in E$, $\lambda_e : Z(e)^* \to Z(E_e)^*$ is surjective.*

Then any locally inner automorphism of A is inner.

Proof. Let α be a locally inner automorphism of A and let $u, v : E \to A$ be maps so that $u(e) \in \alpha(e)Ae$, $v(e) \in eA\alpha(e)$ and $\alpha(x) = u(e)xv(e)$ for every $e \in E$ and $x \in eAe$. By Axiom of Choice and (2), for every non-minimal $e \in E$ there is a map $\mu_e : Z(E_e)^* \to Z(e)^*$ such that $\lambda_e\mu_e = 1_{Z(E_e)^*}$. If e is a minimal element, we consider $Z(E_e)^*$ as a singleton, say $Z(E_e)^* = \{\delta\}$, and we choose μ_e by $\mu(\delta) = e$. Consider the class map $\mathcal{L} : E \times \mathcal{V} \to \mathcal{V}$ given by

(A) $\mathcal{L}(e, \emptyset) = (u(e), v(e))$;

(B) $\mathcal{L}(e, X) = (u(e)\mu_e((v(e)U(f))_{f\in E_e}), \mu_e((v(e)U(f))_{f\in E_e})^{-1}v(e))$ if

$$X = \{(f, (U(f), V(f)) \mid f \in E_e\}$$

and the following conditions hold for every $f \in E_e$

B1 $(v(e)U(f))_{f\in E_e} \in Z(E_e)^*$

B2 $U(f) \in \alpha(f)Af$, $V(f) \in fA\alpha(f)$ and

B3 $\alpha(x) = U(f)xV(f)$ for every $x \in fAf$ and

(C) $\mathcal{L}_1(e, X) = \mathcal{L}_2(e, X) = \emptyset$ otherwise.

By the Recursion Theorem for Well Founded Sets there is a unique map $w : E \to \mathcal{V}$ such that

$$w(e) = \mathcal{L}(e, \{(f, (w(f)) \mid f \in E_e\})$$

for every $e \in E$.

We claim that there are maps $u', v' : E \to A$ such that the following statements hold for every $e \in E$

1. $w(e) = (u'(e), v'(e))$,
2. $(v(e)u'(f))_{f\in E_e} \in Z(E_e)^*$ if e is not minimal,
3. $u'(f) \in \alpha(f)Af$, $v'(f) \in fA\alpha(f)$ and

4. $\alpha(x) = u'(f)xv'(f)$ for every $x \in fAf$.

We prove this by induction on e for well founded sets. If e is minimal in E, then $w(e) = \mathcal{L}(e, \emptyset) = (u(e), v(e))$, so setting $u'(e) = u(e)$ and $v'(e) = v(e)$ conditions 3 and 4 hold and condition 2 vanishes. Now let e be a non-minimal element of E and assume that $w(f) = (u'(f), v'(f))$ for every $f \in E_e$ and conditions 1-4 hold for all the elements in E_e. If $f \in E_e$ then setting $u = \alpha(f)u(e)f, v = fv(e)\alpha(f), u' = u'(f)$ and $v' = v'(f)$, it is easy to see that the following conditions hold: $uv = u'v' = \alpha(f)$, $vu = v'u' = f$ and $\alpha(x) = uxv = u'xv'$ for every $x \in fAf$. By Lemma 2.2 $v(e)u'(f)$ is a central unit of fAf. Moreover for every $f_1 < f < e$, one has $v(e)u'(f)f_1 = v(e)u'(f_1)$. We conclude that $(v(e)u'(f))_{f\in E_e} \in Z(E_e)^*$. Therefore $\{(f, w(f)) \mid f \in E_e\} = \{(f, (u'(f), v'(f))) \mid f \in E_e\}$ is of the form B and satisfies conditions B1-B3 for $U = u'$ and $V = v'$. Therefore

$$w(e) = \mathcal{L}(e, \{(f, w(f)) \mid f \in E_e\}) = (u(e)\mu_e((v(e)u'(f))_{f\in E_e}), \mu_e((v(e)u'(f))_{f\in E_e})^{-1}v(e)).$$

Now set

$$\begin{aligned} u'(e) &= u(e)\mu_e((v(e)u'(f))_{f\in E_e})\text{and} \\ v'(e) &= \mu_e((v)u'(f))_{f\in E_e})^{-1}v(e). \end{aligned}$$

Conditions 1, 3 and 4 follow immediately. Finally, 2 follows from the fact that for every $f \in E_E$, $v(e)u'(f) \in Z(f)$ and the following computation

$$\begin{aligned} u'(e)f &= u(e)\mu_e((v(e)u'(g))_{g\in E_e})f \\ &= u(e)\pi_f(\lambda_e\mu_e((v(e)u'(g))_{g\in E_e})) \\ &= u(e)\pi_f((v(e)u'(g))_{g\in E_e}) \\ &= u(e)v(e)u'(f) = u'(f), \end{aligned}$$

where the second equality is a consequence of applying the equality $\pi_f(\lambda_e(x)) = xf$ to $x = \mu_e((v(e)u'(g))_{g\in E_e})$. So u' and v' satisfy the conditions of Definition and Theorem 1.1 (recall that conditions (a) and (b) of the first statement in Definition and Theorem 1.1 are equivalent) and we conclude that α is inner. ∎

It is desirable to have a sufficient condition that guarantees every locally inner automorphism is inner without the condition that a set of local units E is well founded. If every set of local units which satisfies the condition (2) of Theorem 2.4 also contains a well founded set of local units satisfying condition (2), then condition (1) of Theorem 2.4 would be unnecessary. Unfortunately, when passing from a set of local units E to a well founded set of local units E' contained in E, one has that $Z(E_e)^*$ can be strictly contained in $Z(E'_e)^*$ and hence condition (2) does not pass from E to E'. The following Corollary gives an alternative sufficient condition for every locally inner automorphism being inner.

Note that if E is a set of local units and $e \in E$, then E contains a set of local units E' so that e is the unique minimal element of E'.

Corollary 2.5 *If A has a set of local units E with a unique minimal element e so that for every $f \in E$, the map $\rho_f : Z(f) \to Z(e)$, $\rho_f(x) = xe$, is an isomorphism, then every locally inner automorphism of A is inner.*

Proof. If E_1 is a well founded set of local units contained in E, then $E_1 \cup \{e\}$ is a well founded set of local units satisfying the hypothesis. Thus by Lemma 2.1 we can assume that E is well founded.

Let $e \neq f \in E$ and let $\pi : Z(E_f) \to Z(e)$ denote the projection on the e-th coordinate. Then $\rho_f = \pi \circ \lambda_f$ and so π is surjective. If $\pi((a_{f'})_{f' \in E_e}) = 0$, then $\rho_{f'}(a_{f'}) = 0$ for every $f' \in E_f$. Since ρ_f is an isomorphism,$(a_{f'})_{f' \in E_e} = 0$. This shows that π is a bijection and hence so is λ_f Therefore the restriction of λ_f to $Z(e)^* \to Z(E_e)^*$ is also a bijection. ▮

There are many examples of rings with local units that satisfy the hypotheses of either Theorem 2.4 or Corollary 2.5. We indicate some of these examples next.

Notation 2.6 Matrix rings. Given a set X and a ring R, we adopt the following notation:

$$\begin{aligned}
\mathrm{FM}_X(R) &= \text{Ring of finite matrices over } R \text{ indexed by } X,\\
\mathrm{RFM}_X(R) &= \text{Ring of row finite matrices over } R \text{ indexed by } X,\\
\mathrm{RCFM}_X(R) &= \text{Ring of row and column finite matrices over } R \text{ indexed by } X.
\end{aligned}$$

Incidence rings. Given a preordered set P, a unital ring R and a two-sided ideal J of R, let

$$\begin{aligned}
\mathrm{FI}(P,R,J) &= \{\alpha \in \mathrm{FM}_P(R) \mid x \not\leq y \Rightarrow \alpha(x,y) \in J\}\\
\mathrm{RCFI}(P,R,J) &= \{\alpha \in \mathrm{RCFM}_P(R) \mid x \not\leq y \Rightarrow \alpha(x,y) \in J\}.
\end{aligned}$$

Then $\mathrm{FI}(P,R,J)$ is a subring with local units of $\mathrm{FM}_P(R)$. If $J = 0$, then $FI(P,R) = FI(P,R,0)$ is the incidence ring of P over R.

Let X be a set and $i,j \in X$. Then e_{ij} is going to denote the matrix indexed by X having 1 at the (i,j)-entry and 0 elsewhere and $e_i = e_{ii}$. The set X will be always clear from the context. If Y is a subset of X, then $e_Y = \sum_{i \in Y} e_i$. Note that the sum makes sense even if Y is infinite.

If we are working with a direct product $\prod_{i \in X} R_i$ of rings, the meaning of e_i and e_Y, for $i \in X$ and $Y \subseteq X$ will be different but without possible confusion. In that situation, e_i is the X-tuple having 1 at the i-coordinate and 0 elsewhere and $e_Y = \sum_{i \in Y} e_i$.

We refer the reader to [3] for the definition and notation about smash product rings.

Corollary 2.7 *For each of the following rings, every locally inner automorphism is inner, and so every piecewise inner automorphism is inner.*

1. *The direct sum of rings with identity.*
2. *The ring $BRFM_P(R)$ of row finite matrices indexed by an infinite set P, having entries from a unital ring such that the cardinality of the set of their nonzero entries is strictly smaller than the cardinality of P.*
3. *$FI(P,R,J)$ where R is a ring with unit, P is a preordered set and J is an ideal of R. This includes the ring $FM_P(R)$ of finite matrices over R indexed by a set P.*
4. *The path algebra of a quiver with infinitely many vertices. That is the non unital algebra generated by the oriented paths of the quiver. (Note that this algebra is unital if and only if the set of vertices of the quiver is finite).*

5. *The smash product of a strongly graded ring.*

6. *The smash product of a polynomial ring in several (even infinite) variables with coefficients on a unital ring R (graded by the free abelian group generated by the variables).*

Proof. (1) Is a particular case of (3) taking I as a partially ordered set with no non trivial relations. However we include a short and less technical proof for this. Let $A = \oplus_{i\in I} R_i$, a direct sum of unital rings. The set $E = \{e_X : X \text{ finite subset of } I\}$ is a well founded set of local units. Moreover, if X is a finite subset of I, then $E_{e_X} = \{e_Y : Y \subset X\}$ and $Z(e_X) = \oplus_{i\in X} Z(R_i)$. If X has just one element, then $Z(E_{e_X}) = \{0\}$ and λ_{e_X} is surjective. Assume now that X has more than one element and let $a = (a_Y)_{Y\subset X} \in Z(E_{e_X})$. Then the element $b = (b_i) \in e_X A e_X$, defined via $b_i = 0$ if $i \notin X$ and $b_i = a_{\{i\}}$ if $i \in X$, belongs to $Z(e_X)$ and $\lambda_{e_X}(b) = a$. So the hypothesis of Corollary 2.5 holds.

(2) Let $A = \mathrm{BRFM}_P(R)$ and fix an element $x \in X$. Then $\{e_Y : x \in Y, \mid Y \mid < \mid X \mid\}$ is a set of local units of A which satisfies the conditions of Corollary 2.5.

(3) For every subset X of P we define the following equivalent relation $\sim_X$ on X.

$$\begin{aligned} x \sim_X y \quad \Leftrightarrow \quad & (\exists a_0, a_1, \dots, a_n \in X) x = a_0, y = a_n, \\ & (\forall i = 1, \dots n)(a_{i-1} \leq a_i \text{ or } a_{i-1} \geq a_i) \end{aligned}$$

(Note that $\sim_X$ depends on the set X). Then,

$$\begin{aligned} Z(e_X) = \{\alpha \in A : \quad & \alpha(x,y) \neq 0 \Rightarrow x = y \in X, \\ & \alpha(x,x) \in Z(R), \\ & \alpha(x,x) - \alpha(y,y) \in \mathrm{Ann}_{Z(R)}(J), \\ & \text{and } x \sim_X y \Rightarrow \alpha(x,x) = \alpha(y,y)\} \end{aligned}$$

(where $\mathrm{Ann}_{Z(R)}(J)$ denotes the annihilator of J in $Z(R)$. We leave it to the reader to check the details.

Let $\sim$ denote the equivalent relation on P given by $\sim = \cup_{|X|<\infty} \sim_X$. Let Y be a set of representatives of $X/\sim$. Then

$$\begin{aligned} E = \{e_X : \quad & X \subseteq P, \mid X \mid < \infty, (\forall x \in X)(\exists a_0, a_1, a_2, \dots, a_n \in X) x = a_0, a_n \in Y, \\ & \text{and } (\forall i = 1, \dots n)(a_{i-1} \leq a_i \text{ or } a_{i-1} \geq a_i)\} \end{aligned} .$$

is a set of local units of A. We prove that E satisfies the conditions of Theorem 2.4. Plainly E is well founded. Note that for every finite subset X of P, there exist $x = y_0^x, y_1^x, \dots, y_{n_x}^x \in P$ such that $y_{n_x}^x \in Y$ and for every $i = 1, \dots, n_x$, either $y_{i-1}^x \leq y_i^x$ or $y_{i-1}^x \geq y_i^x$. Thus $X \subseteq \hat{X} = \cup_{x\in X}\{y_0^x, y_1^x, \dots, y_{n_x}^x\}$ and $e_{\hat{X}} \in E$. Now we check condition (2) of Theorem 2.4. If $e_X \in E$, then the map $\rho_X : Z(e_X) \to Z(e_{Y\cap X})$, given by $\rho_X(a) = a e_Y$, is a ring isomorphism. Moreover, $\rho_X = \pi_{Y\cap X} \circ \lambda_e$ where $\pi_{Y\cap X} : Z(E_e) \to Z(e_{Y\cap X})$ is the projection. As in the proof of Corollary 2.5 one can prove that π_Y is an isomorphism and hence λ_e is an isomorphism for every $e \in E$. It only remains to apply Theorem 2.4.

Statement (4) follows by similar arguments as those found in (3).

(5) (See [3] for the notation). Let G be a group with identity 1 and R a strongly graded ring. Set $E = \{e_X : 1 \in X \subseteq G, \mid X \mid < \infty\} \subseteq R\#P_G$. Let X be a finite subset of G which

contains 1. For every g in G let $x_g^1, x_g^2, \ldots, x_g^{n_g} \in R_g$ and $y_{g^{-1}}^1, y_{g^{-1}}^2, \ldots, y_{g^{-1}}^{n_g} \in R_{g^{-1}}$ such that $\sum_{i=1}^{n_g} x_g^i y_{g^{-1}}^i = 1$. By straightforward arguments one can show that

$$Z(e_X) = \left\{ \alpha \in e_X R \# P_G e_X : \alpha(g,g') = \begin{cases} 0 & \text{if } g \neq g' \\ \sum_{i=1}^{n_g} x_g^i \alpha(1,1) y_{g^{-1}}^i & \text{if } g = g' \in X. \end{cases} \right\}$$

Then it is clear that the map $\rho_X : Z(e_X) \to Z(e_1)$ is an isomorphism and Corollary 2.4 applies.

(6) This can be seen as a particular case of (3) because such a smash product is isomorphic to the incidence ring $\mathrm{FI}(G,R)$, where G is the group generated by the variables with the order given by $x \leq y$ if and only $x^{-1}y$ belongs to the monoid generated by the variables.

3 Counterexamples.

Having found a class of rings for which inner and locally inner automorphisms are the same, we turn our attention to proving that this is not the case in general. As well as giving one example of a piecewise inner automorphism that is not inner, we give an example of an inner automorphism which is not piecewise defined.

Example 3.1 An inner (and so locally inner) automorphism which is not piecewise defined (and so not piecewise inner). Let R be an arbitrary unital ring and $A = \mathrm{FM}_{\mathbf{Z}}(R)$. Let $u \in \mathrm{RCFM}_{\mathbf{Z}}(R)$ be given by $u(x,y) = \delta_{x+1,y}$. Then u is invertible and conjugation by u restricts to an inner automorphism of A. But it is not difficult to see that 0 is the only α-invariant element of A.

Example 3.2 A piecewise inner automorphism that is not inner. Let K be any unital ring which admits a non-inner automorphism and let ϕ be such. Let

$$A_1 = K[X_1, X_1^{-1}, X_2, X_2^{-1}, \ldots]$$

be the ring of skew Laurent polynomials in countably many commuting variables, so that $X_n k = \phi(k) X_n$ for every $n \in \mathbf{N}$ and $k \in K$. For every $n \in \mathbf{N}$, let $A_n = K[X_n, X_n^{-1}, \ldots]$ be the subring of A_1 generated by $K \cup \{X_m, X_m^{-1} : m \geq n\}$.

Let A be the ring of upper triangular matrices α indexed by $\mathbf{N}$, such that for every $n, m \in \mathbf{N}$, the (n,m)-entry $\alpha(n,m)$ of α belongs to A_n and $\alpha(n,m) = 0$ for almost all $(n,m) \in \mathbf{N}^2$. For every $n \in \mathbf{N}$, let $f_n = \sum_{i=1}^{n} e_i$. Then $\{f_n \mid n \in N\}$ is a set of local units of A. Let Φ be the automorphism of A given by the following condition: For every $n \in \mathbf{N}$, the restriction $\Phi|_{Af_n}$ of Φ to Af_n is conjugation by $u_n = X_n f_n$. (Note that $Af_n = f_n A f_n$). Φ is well-defined because $X_n^{-1} f_n \cdot X_{n+1} f_{n+1}$ is central in Af_n for every $n \in \mathbf{N}$. And by definition, Φ is piecewise inner.

Assume that Φ is inner. By Definition 1.1, for every $n \in \mathbf{N}$, there are $u'_n \in Af_n$ and $v'_n \in Af_n$ such that if $x \in Af_n$, $\Phi(x) = u'_n x v'_n$ and $u'_{n+1} f_n = u'_n$. By Lemma 2.2, for every $n \in \mathbf{N}$ there are $c_n, d_n \in Z(f_n)$ such that $u'_n = u_n d_n$, $u'_n = c_n u_n$ and $c_n d_n = f_n = d_n c_n$. Then $d_n = \lambda_n f_n$ for some invertible element $\lambda_n \in A_n$. Thus, $u'_1 = u'_n f_1 = u_n d_n f_1 = \lambda_n X_n f_1$ and $v'_1 = v'_n f_1 = c_n v_n f_1 = \lambda_n^{-1} X_n f_1 \in A_n f_1$ for every $n \in \mathbf{N}$. Therefore $\lambda_1 \in X_1^{-1} A_n$ for every n and hence $u'_1 = a f_1$ and $v_1 = a^{-1} f_1$ for some $a \in K$. We conclude that $\phi(x) = axa^{-1}$ for every $x \in K$ and this yields to a contradiction with the assumption on ϕ.

4 Skolem-Noether results.

Several versions of the Skolem-Noether Theorem exist for finitely generated algebras (see [10], [7] and [8]). In this section we show how Theorem 2.4 can be applied to obtain a Skolem-Noether result for certain classes of infinitely generated algebras. As we mentioned in the introduction, this is an example of how non-unital results can be used to obtain results about unital rings.

Let R be a unital ring, P a partially ordered (not necessarily locally finite) and $A = \mathrm{FI}(P, R)$, the incidence ring. Then the ring of multipliers $Q(A)$ of A can be identified with $\mathrm{RCFI}(P, R)$. For example if the order in P is given by $x \leq y$ for every $x, y \in P$, then $\mathrm{FI}(P, R, J) = \mathrm{FM}_P(R)$ and its ring of multipliers is $\mathrm{RCFM}_P(R)$. If P is totally ordered, then $\mathrm{RCFI}(P, R)$ can be viewed as an upper triangular matrix ring.

Recall from [1] that every automorphism of a ring with local units A extends uniquely to an automorphism of $Q(A)$ and an automorphism of A is inner if and only if it is the restriction of a inner automorphism of $Q(A)$. Moreover in some cases every automomorphism of $Q(A)$ restricts to an automorphism of A; see [9] and [1].

Given a unital ring A, a subring R of A and a set X we consider R embedded in $\mathrm{RCFM}_X(A)$ diagonally. In particular, if Y is a finite subset of X and R is a unital subring of A, then $R + M_Y(A)$ is a unital subring of $\mathrm{RCFM}_X(A)$.

Theorem 4.1 *Let A be a ring and R a subring of A. Assume that for every pair of finite sets X and $Y \subseteq X$, every injective R-homomorphism of algebras $R + M_Y(A) \to M_X(A)$ extends to an inner automorphism of $M_X(A)$. Then all the R-automorphisms of the rings RCFM$_X(A)$ and all the R-automorphisms of RFM$_X(A)$ are inner.*

Proof. Let α be an R-automorphism of $B = \mathrm{RCFM}_X(A)$. By [9] α restricts to an automorphism of $I = \mathrm{FM}_X(A)$. Let α' be the restriction of α to an automorphism of I. We claim that α' is locally inner. The set $Y = \{e_F : F \text{ finite subset of } X\}$ is a set of local units of I. For every finite subset F of X let $I_F = R + e_F I e_F \simeq R + M_F(A)$ and F' be a finite subset of X such that $e_F, \alpha(e_F) \leq F'$. Then α restricts to an injective R-homomorphism $\alpha_F : I_F \to I_{F'}$. By the hypothesis and the arguments given prior to the theorem, there exists an invertible element w of $I_{F'}$, such that $\alpha'(x) = \alpha_F(x) = w^{-1}xw$, for every $x \in I_F$. Thus α' is inner at e_F for every finite subset F of X and hence α' is locally inner. By Corollary 2.7, α' is inner. Since B is the ring of multipliers of I, α' is the restriction of an inner automorphism β of B and every automorphism of I extends uniquely to an automorphism of B (see [1]). Thus $\alpha = \beta$ is inner.

Now assume that α is an R-automorphism of $\mathrm{RFM}_X(A)$. By [9, Proposition 6], there is an inner automorphism τ of $\mathrm{RFM}_X(A)$ such that $\alpha\tau$ restricts to an automorphism of I. Thus $\alpha\tau$ restricts to an automorphism of B. By the previous paragraph this automorphism is inner over B. Since every automorphism of I extends uniquely to an automorphism of $\mathrm{RFM}_X(R)$, $\alpha\tau$ is inner over $\mathrm{RFM}_X(A)$ and hence α is inner over $\mathrm{RFM}_X(A)$. ∎

Note that by the classical Skolem-Noether Theorem, every central simple algebra satisfies the conditions of Theorem 4.1. More generally, if S is a semilocal ring and A is a separable finitely generated projective S-algebra such that the centre R of A has no idempotents other than 0 and 1, then the conditions of Theorem 4.1 holds. This is a direct consequence of a

Theorem of Childs and DeMeyer [8, Theorem 1.2]. Thus combining Theorem 4.1 and the Childs and DeMeyer Theorem one deduces the following corollary.

Corollary 4.2 *Let S be a semilocal ring and A a separable finitely generated projective S-algebra whose centre R of A has no idempotents other than* 0 *and* 1*. Then every K-automorphism of the ring $RCFM_X(A)$ of row and column finite matrices (resp. the ring $RFM_X(A)$ of row finite matrices) over A indexed by a set X is inner.*

Jøndrup has given several Skolem-Noether type Theorems for triangular matrix rings [10] [11]. It is natural to expect to obtain results similar to Corollary 4.2 for infinite triangular matrix rings by using Jøndrup's results and arguments as above. Unfortunately the more general statement does not hold. Indeed, let R be a unital ring, $X = \mathbf{Z}$ and α the automorphism of the RCFI$(\mathbf{Z}, R)$ induced by the the automorphism of X given by $n \mapsto n+1$. Then α is a noninner R-automorphism. However, using an argument similar to the one found in the proof of Theorem 4.1 together with the above mentioned results from [10] and [11], we can prove the final three results of this paper. We leave the details to the reader.

Theorem 4.3 *Let R be a artinian simple algebra finite dimensional over its center K, X a totally ordered set and $A = RCFI(X, R)$ the incidence ring of X over R. Let α be a K-automorphism of A such that the set $\{e_F \mid F$ is finite$\}$ contains a set of α-invariant local units of $FI(X, R)$. Then α is inner.*

Theorem 4.4 *Let K be a commutative ring, R a prime K-algebra and X a totally ordered set. Let α be an R-automorphism of $RCFI(X, R)$, such that $\{e_F \mid F$ is finite$\}$ contains a set of α-invariant local units of $FI(X, R)$. Then α is inner.*

Theorem 4.5 *Let R-be a commutative factorial domain, P a prime ideal of R and X be a totally ordered set. Then every R-automorphism of $RCFI(X, R, P)$ such that $\{e_F \mid F$ is finite$\}$ contains a set of α-invariant local units of $FI(X, R, P)$ is inner.*

References

[1] G. Abrams, J. Haefner and Á. del Río, Approximating rings with local units via automorphisms, Acta Math. Hungarica 82 (3) (1999), 207–226.

[2] P.N. Ánh and L. Márki, "Morita equivalence for rings without identity", *Tsukuba J. Math* **11**(1) (1987), 1–16.

[3] M. Beattie, A generalization of the smash product of a graded ring, *J. Pure Appl. Algebra* **52** (1988), 219–226.

[4] M. Beattie and Á. del Río, The Picard group of a category of graded modules, *Comm. in Algebra* **24** (1996), 4397–4414.

[5] R. Bonnet and M. Rubin, On well-generated Boolean algebras. Ann. Pure Appl. Logic 105 (2000), 1–50.

[6] R. Bonnet and M. Rubin, On essentially low, canonically well-generated Boolean algebras. J. Symbolic Logic 67 (2002), 369–396.

[7] S.U. Chase, Two remarks on central simple algebras, Comm. Algebra, 12(18), (1984) 2279–2289.

[8] L.N. Childs and F.R. DeMeyer, On automorphisms of separable algebras, Pacific J. Math. **23**, (1967) 25–34.

[9] J. Haefner, Á. del Río, J.J. Simón, Isomorphisms of row and column finite matrix rings, *Proc. Amer. Math. Soc.*, **125** (1997), 1651–1658.

[10] S. Jøndrup, Automorphisms of upper triangular matrix rings, *Arch. der Math.* **49** (1987), 497–502.

[11] S. Jøndrup, The group of automorphisms of certain subalgebras of matrix algebras, *J. Algebra* **141** (1991), 106–114.

[12] K. Kunen, *Set Theory. An Introduction to Independence Proofs*, Studies in Logic and the Foundations of Mathematics 102, North-Holland, Amsterdam, 1980.

[13] M. Petrich, Ideal extensions of rings, Acta. Math. Hungarica 45 (1985) 263–285.