Presentation of the group of units of $\mathbb{Z}D_{16}^{-}$

Antonio Pita

December 28, 2004

Abstract

We obtain a presentation of the group of units of the integral group ring of the group $D_{16}^{-} = \langle a, b | a^8 = 1 = b^2, ba = a^3 b\rangle$.

One of the Open Problems listed by Sehgal in [9] asks for obtaining presentations by generators and relations of the group of units $\mathbb{Z}G^*$ of the integral group ring $\mathbb{Z}G$ for some finite groups G. This problem has been solved up to finite index for seven of the nine non abelian groups of order 16. (Recall that Higman describes the structure of $\mathbb{Z}G^*$ for G abelian [4].) Namely it is well known since Higman that $\mathbb{Z}(Q_8 \times C_2)^* = Q_8 \times C_2$ [9] (see notation below). If G is $D_8 \times C_2$, $C_4 \rtimes C_4$, Q_{16} or $\mathcal{H}$ then $\mathbb{Z}G^*$ contains a subgroup of finite index which is a direct product of free groups (see [5] and [6]). Furthermore a concrete direct product of free groups F of $\mathbb{Z}G^*$ with minimal index in $\mathbb{Z}G^*$ has been computed in [8]. A presentation of $\mathbb{Z}G^*$ has been obtained in [7] for the groups $\mathcal{D}$ and D_{16}^{+}. Thus at this moment there are two groups of order 16 for which no presentation of $\mathbb{Z}G^*$ by generators and relations is known, namely these groups are D_{16} and D_{16}^{-}.

$$\begin{array}{rcl}
C_n & = & \text{Cyclic Group of order n.} \\
Q_{4n} & = & \text{Quaternion group of order 4n.} \\
D_{2n} & = & \text{Dihedral group of order 2n.} \\
C_4 \rtimes C_4 & = & \langle a, b : a^4 = b^4 = 1, ba = a^{-1}b\rangle. \\
\mathcal{H} & = & \langle a, b : a^4 = b^4 = (ab)^2 = (a^2, b) = 1\rangle. \\
\mathcal{D} & = & \langle a, b, c : a^2 = 1 = b^2, c^4 = 1, ba = c^2ab, c \text{ central}\rangle. \\
D_{16}^{+} & = & \langle a, b : a^8 = 1 = b^2, ba = a^5 b\rangle. \\
D_{16}^{-} & = & \langle a, b : a^8 = 1 = b^2, ba = a^3 b\rangle
\end{array}$$

In this paper we obtain the presentation of the group of units of $\mathbb{Z}D_{16}^{-}$. More precisely we prove the following theorem.

Theorem 1 *$\mathbb{Z}D_{16}^{-} \simeq (\Gamma \rtimes F_3) \rtimes (\pm D_{16}^{-})$, where F_3 is a free group of rank 3 and Γ is given by the following presentation*

$$\Gamma = \left\langle \begin{array}{l} A_0, A_1, B_0, B_1, C, D, I, J, \\ E_0, E_1, F_0, F_1, G_0, G_1, H_0, H_1, \\ K_0, K_1, K_2, K_3, L_0, L_1, M_0, M_1, \\ N_0, N_1, P_0, P_1, Q_0, Q_1, R_0, R_1, \\ S, T \end{array} \left| \begin{array}{lclcl} F_0^{-1}K_2^{-1}JK_0 & = & F_1^{-1}K_3^{-1}J^{-1}K_1 & = & \\ E_0^{-1}K_2^{-1}IK_0 & = & E_1^{-1}K_3^{-1}I^{-1}K_1 & = & \\ N_0^{-1}K_2^{-1}M_0K_0 & = & N_1^{-1}K_3^{-1}M_1K_1 & = & \\ P_0^{-1}K_2^{-1}R_0K_0 & = & P_1^{-1}K_3^{-1}R_1K_1 & = & \\ [S, T] & = & X_0^{-1}X_1'X_1^{-1}X_0' & = & 1 \end{array} \right. \right\rangle \quad (1)$$

where:

$$X_i' = \begin{cases} S & \textit{if} \quad X_i = A_i \textit{ or } B_i \\ T & \textit{if} \quad X_i = E_i \textit{ or } F_i \\ A_i & \textit{if} \quad X_i = F_i, H_i \textit{ or } N_i \\ B_i & \textit{if} \quad X_i = H_i, M_i \textit{ or } J \\ C & \textit{if} \quad X_i = E_i, G_i \textit{ or } P_i \\ D & \textit{if} \quad X_i = G_i, R_i \textit{ or } I \\ L_i & \textit{if} \quad X_i = M_i, R_i, H_i \textit{ or } G_i \\ Q_i & \textit{if} \quad X_i = N_i, P_i, H_i \textit{ or } G_i \end{cases}$$

A big part of the proof of Theorem 1 is based in a result of Jespers and Parmenter [5] who proved that $\mathbb{Z}D_{16}^- = (\Gamma \rtimes F_3) \rtimes (\pm D_{16}^-)$ where Γ is the subgroup of $\mathrm{SL}_2(\mathbb{Z}[i])$ formed by the matrices of the form

$$I + \begin{pmatrix} 4m + 2m'\sqrt{-2} & 4n + 4n'\sqrt{-2} \\ 2e + e'\sqrt{-2} & 4f + 2f'\sqrt{-2} \end{pmatrix} \tag{2}$$

with $m, m', n, n', e, e', f, f' \in \mathbb{Z}$. Thus we only have to show that the group Γ has a presentation as in (1). In order to obtain such a presentation, we use that Γ is Kleinian and hence we can apply Poincaré's Method to obtain a presentation of Γ from a fundamental polyhedron (see [1],[2] or [3]). Recall that a Kleinian group is a group that acts discontinuously on the 3-dimensional hyperbolic space by the Poincaré extension of the action of $\mathrm{PSL}_2(\mathbb{C})$ by Möbius Transformations.

So the first step is obtaining a fundamental polyhedron of the action of Γ on $\mathbb{H}^3$. We achieve this goal by computing a variation of Ford's fundamental polyhedron [1], that is, we compute the fundamental polyhedron of Γ formed by the intersection of the fundamental polyhedron of Γ_∞, the subgroup of Γ formed by the elements that stabilize ∞, and the exterior of the isometric half spheres of the elements of $\Gamma \setminus \Gamma_\infty$. We use Poincaré's model of $\mathbb{H}^3$ of the hyperbolic 3-dimensional space, that is

$$\mathbb{H}^3 = \{(x, y, r) \in \mathbb{R}^3 : r > 0\}$$

and the action by isometries of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{H}^3$ is as explained in [2]. Since

$$\Gamma_\infty = \langle \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4\sqrt{-2} \\ 0 & 1 \end{pmatrix} \rangle$$

a fundamental polyhedron of Γ_∞ is the infinite strip $F_\infty = [-2, 2] \times [-2\sqrt{2}, 2\sqrt{2}] \times \mathbb{R}^+$.

Recall that the isometric circle of an element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$, with $c \neq 0$ is the circle given by the equation $|cz + d| = 1$, that is the circle with centre $-d/c$ and radius $1/|c|$. The isometric half sphere of g (as above) is the (euclidean) half sphere of $\mathbb{H}^3$ with the same centre and radius as the isometric circle of g.

Remark 2 *Notice that an isometric circle is in fact a circumference and not a circle. We use this notation to keep the most extended terminology. By the inside part of an isometric circle we mean the open circle limited by it and the exterior part is the complement of the inside part. The inside and the exterior part of an isometric half sphere are defined similarly.*

We need a technical lemma:

Lemma 3 *If $x \in \mathbb{C}$ is of one of the following forms:*

$$\left. \begin{array}{c} a + b\sqrt{-2} \\ (\frac{i}{3} + 2a) + (\frac{j}{3} + b)\sqrt{-2} \\ 1 + 2a + (\frac{1}{2} + b)\sqrt{-2} \end{array} \right\} \textit{with } a, b \in \mathbb{Z} \textit{ and } i, j = \pm 2 \tag{3}$$

then x is not inside the isometric circle of an element of Γ*.*

Proof. Consider an element $g \in \Gamma$ of the form (2), and its isometric sphere S, with centre $c = -\frac{1+4f+2f'\sqrt{-2}}{2e+e'\sqrt{-2}}$ and radius $r = \frac{1}{\|2e+e'\sqrt{-2}\|}$. Let S_0 denote the interior part of S.

Assume first that $x = a + b\sqrt{-2}$ with $a, b \in \mathbb{Z}$. Then

$$\begin{aligned} \|c - x\| = \; & r\| - (1 + 4f + 2f'\sqrt{-2}) - (a + b\sqrt{-2})(2e + e'\sqrt{-2})\| \\ = \; & r\| - 1 - 4f - 2ae + 2be' - (2f' + ae' + 2be)\sqrt{-2}\|. \end{aligned}$$

If $x \in S_0$ and $y = -1 - 4f - 2ae + 2be' - (2f' + ae' + 2be)\sqrt{-2}$ then $\|y\| < 1$ and hence $y = 0$ which is not possible.

Assume now that $x = (\frac{i}{3} + 2a) + (\frac{j}{3} + b)\sqrt{-2}$ with $a, b \in \mathbb{Z}$ and $i, j = \pm 2$. Then $\|c - x\| = \frac{r}{3}\|y\|$ where

$$y = -3 - 12f - 2ei - 12ae + 2je' + 6be' + (-6f' - ie' - 6ae' - 2je - 6be)\sqrt{-2}$$

and thus $x \in S_0$ if and only if $\|y\| < 3$. Figure 1 represents the elements of $\mathbb{Z}[\sqrt{-2}]$ of norm at most 3. Write $y = y_1 + y_2\sqrt{-2}$ with $y_1 = -3-12f-2ei-12ae+2je'+6be'$ and $y_2 = -6f'-ie'-6ae'-2je-6be$. Since y_1 is odd and y_2 is even, because i is even, a glance at Figure 1 shows that $y = \pm 1$. Thus $y_2 = 0$ and hence 6 divides $ie' + 2je$. This implies that if $i = j$ then $0 \equiv e' + 2e \equiv -e' - 2e \equiv 2e' - 2e \mod 3$ and if $i = -j$ then $0 \equiv e' - 2e \equiv -e' + 2e \equiv 2e' + 2e \mod 3$, a contradiction in either case.

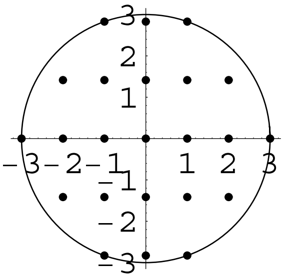

Figure 1

Finally assume that $x = 1 + 2a + (\frac{1}{2} + b)\sqrt{-2}$ with $a, b \in \mathbb{Z}$. Then $\|c - x\| = r\|y\|$ where

$$y = -1 - 4f - 2e - 4ae + e' + 2be' - (2f' + e' + 2ae' + e + 2be)\sqrt{-2}$$

and $x \in S_0$ if and only if $y = 0$ if and only if $x = c$. Write $g = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ with $x_{ij} \in \mathbb{Z}[\sqrt{-2}]$. Then $c = \frac{-x_{22}}{x_{21}} = x$ and hence

$$-2x_{22} = x_{21}[(2 + 4a) + (1 + 2b)\sqrt{-2}]$$

since $\det g = 1$, x_{21} and x_{22} are coprime in $\mathbb{Z}[\sqrt{-2}]$ and therefore x_{21} divides 2 in $\mathbb{Z}[\sqrt{-2}]$, that is x_{21} is ± 1 or ± 2. Since $x_{21} = 2e + e'\sqrt{-2} \neq \pm 1$, we deduce that

$$1 + 4f + 2f'\sqrt{-2} = x_{22} = \pm(2 + 4a + (1 + 2b)\sqrt{-2})$$

a contradiction. ▮

Proposition 4 *Let Γ be the subgroup of $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$ formed by the matrices of the form (2) with $m, m', n, n', e, e', f, f' \in \mathbb{Z}$ and let D be the intersection of $F_\infty = [-2,2] \times [-2\sqrt{-2}, 2\sqrt{-2}] \times \mathbb{R}^+$ with the exterior of the half spheres with centre and radius in one of the cases of the following table*

Centre	*Radius*	f	f'
$\pm(2f' - (\frac{1}{2} + 2f)\sqrt{-2})$	$\frac{1}{\sqrt{2}}$	$-1, 0$	$0, \pm 1$
$\pm(\frac{1}{2} + 2f + f'\sqrt{-2})$	$\frac{1}{2}$	$-1, 0$	$0, \pm 1, \pm 2$
$\pm(f' - (\frac{1}{4} + f)\sqrt{-2})$	$\frac{1}{\sqrt{8}}$	$-2, -1, 0, 1$	± 1
$\pm(\frac{1}{4} + f + \frac{f'}{2}\sqrt{-2})$	$\frac{1}{4}$	± 1	$\pm 1, \pm 3$

Then D is a fundamental polyhedron of the group Γ.

Proof. As mentioned above F_∞ is a fundamental polyhedron of the stabilizer Γ_∞ of ∞ and the intersection D_1 of F_∞ and the exterior of the isometric half spheres of $\Gamma \setminus \Gamma_\infty$ is a fundamental polyhedron of Γ. We want to show that $D = D_1$.

To prove $D_1 \subseteq D$ we show that the half spheres mentioned in the proposition are isometric half spheres of elements of $\Gamma \setminus \Gamma_\infty$. The 12 half spheres of radius $\frac{1}{\sqrt{2}}$ are the isometric half spheres of the following elements

$$\begin{array}{lll} A_0 = \begin{pmatrix} -3-2\sqrt{-2} & 8\sqrt{-2} \\ -\sqrt{-2} & -3+2\sqrt{-2} \end{pmatrix} & B_0 = \begin{pmatrix} 1+2\sqrt{-2} & -4\sqrt{-2} \\ \sqrt{-2} & 1-2\sqrt{-2} \end{pmatrix} & C = \begin{pmatrix} -3 & 4\sqrt{-2} \\ -\sqrt{-2} & -3 \end{pmatrix} \\ A_1 = \begin{pmatrix} -3-2\sqrt{-2} & -8\sqrt{-2} \\ \sqrt{-2} & -3+2\sqrt{-2} \end{pmatrix} & B_1 = \begin{pmatrix} 1+2\sqrt{-2} & 4\sqrt{-2} \\ -\sqrt{-2} & 1-2\sqrt{-2} \end{pmatrix} & D = \begin{pmatrix} 1 & 0 \\ \sqrt{-2} & 1 \end{pmatrix} \end{array} \tag{4}$$

and their inverses. The 20 spheres of radius $\frac{1}{2}$ are the isometric half spheres of the following elements

$$\begin{array}{lll} E_0 = \begin{pmatrix} 1-4\sqrt{-2} & -16 \\ -2 & 1+4\sqrt{-2} \end{pmatrix} & F_0 = \begin{pmatrix} -3+4\sqrt{-2} & 20 \\ 2 & -3-4\sqrt{-2} \end{pmatrix} & I = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \\ E_1 = \begin{pmatrix} 1-4\sqrt{-2} & 16 \\ 2 & 1+4\sqrt{-2} \end{pmatrix} & F_1 = \begin{pmatrix} -3+4\sqrt{-2} & -20 \\ -2 & -3-4\sqrt{-2} \end{pmatrix} & \\ G_0 = \begin{pmatrix} 1-2\sqrt{-2} & -4 \\ -2 & 1+2\sqrt{-2} \end{pmatrix} & H_0 = \begin{pmatrix} -3+2\sqrt{-2} & 8 \\ 2 & -3-2\sqrt{-2} \end{pmatrix} & J = \begin{pmatrix} -3 & 4 \\ 2 & -3 \end{pmatrix} \\ G_1 = \begin{pmatrix} 1-2\sqrt{-2} & 4 \\ 2 & 1+2\sqrt{-2} \end{pmatrix} & H_1 = \begin{pmatrix} -3+2\sqrt{-2} & -8 \\ -2 & -3-2\sqrt{-2} \end{pmatrix} & \end{array} \tag{5}$$

and their inverses. The 16 spheres of radius $\frac{1}{\sqrt{8}}$ are the isometric half spheres of the following elements

$$\begin{array}{ll} K_0 = \begin{pmatrix} 1-2\sqrt{-2} & -8 \\ -2\sqrt{-2} & -7+2\sqrt{-2} \end{pmatrix} & K_2 = \begin{pmatrix} 1+2\sqrt{-2} & 8 \\ -2\sqrt{-2} & -7-2\sqrt{-2} \end{pmatrix} \\ K_1 = \begin{pmatrix} 1-2\sqrt{-2} & 8 \\ 2\sqrt{-2} & -7+2\sqrt{-2} \end{pmatrix} & K_3 = \begin{pmatrix} 1+2\sqrt{-2} & -8 \\ 2\sqrt{-2} & -7-2\sqrt{-2} \end{pmatrix} \\ L_0 = \begin{pmatrix} -3-2\sqrt{-2} & 4\sqrt{-2} \\ -2\sqrt{-2} & -3+2\sqrt{-2} \end{pmatrix} & Q_0 = \begin{pmatrix} 5+2\sqrt{-2} & -8\sqrt{-2} \\ 2\sqrt{-2} & 5-2\sqrt{-2} \end{pmatrix} \\ L_1 = \begin{pmatrix} -3-2\sqrt{-2} & -4\sqrt{-2} \\ 2\sqrt{-2} & -3+2\sqrt{-2} \end{pmatrix} & Q_1 = \begin{pmatrix} 5+2\sqrt{-2} & 8\sqrt{-2} \\ -2\sqrt{-2} & 5-2\sqrt{-2} \end{pmatrix} \end{array} \tag{6}$$

and their inverses. Finally the 16 spheres of radius $\frac{1}{4}$ are the isometric half spheres of the following elements

$$\begin{array}{ll}
M_0 = \begin{pmatrix} 5-2\sqrt{-2} & -8 \\ -4 & 5+2\sqrt{-2} \end{pmatrix} & N_0 = \begin{pmatrix} 5-6\sqrt{-2} & -24 \\ -4 & 5+6\sqrt{-2} \end{pmatrix} \\
M_1 = \begin{pmatrix} 5-2\sqrt{-2} & 8 \\ 4 & 5+2\sqrt{-2} \end{pmatrix} & N_1 = \begin{pmatrix} 5-6\sqrt{-2} & 24 \\ 4 & 5+6\sqrt{-2} \end{pmatrix} \\
P_0 = \begin{pmatrix} -3+6\sqrt{-2} & 20 \\ 4 & -3-6\sqrt{-2} \end{pmatrix} & R_0 = \begin{pmatrix} -3+2\sqrt{-2} & 4 \\ 4 & -3-2\sqrt{-2} \end{pmatrix} \\
P_1 = \begin{pmatrix} -3+6\sqrt{-2} & -20 \\ -4 & -3-6\sqrt{-2} \end{pmatrix} & R_1 = \begin{pmatrix} -3+2\sqrt{-2} & -4 \\ -4 & -3-2\sqrt{-2} \end{pmatrix}
\end{array} \tag{7}$$

and their inverses.

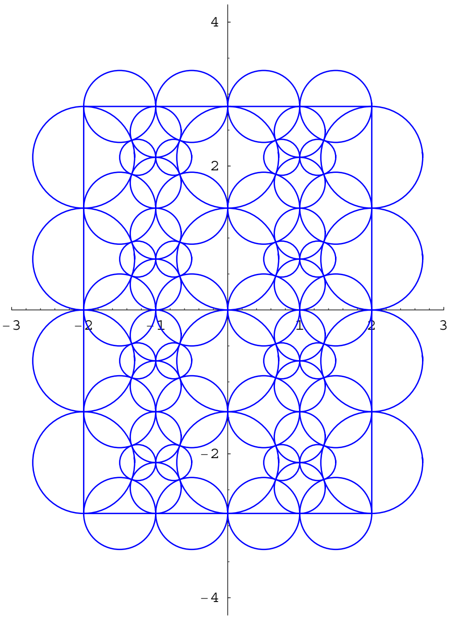

Figure 2

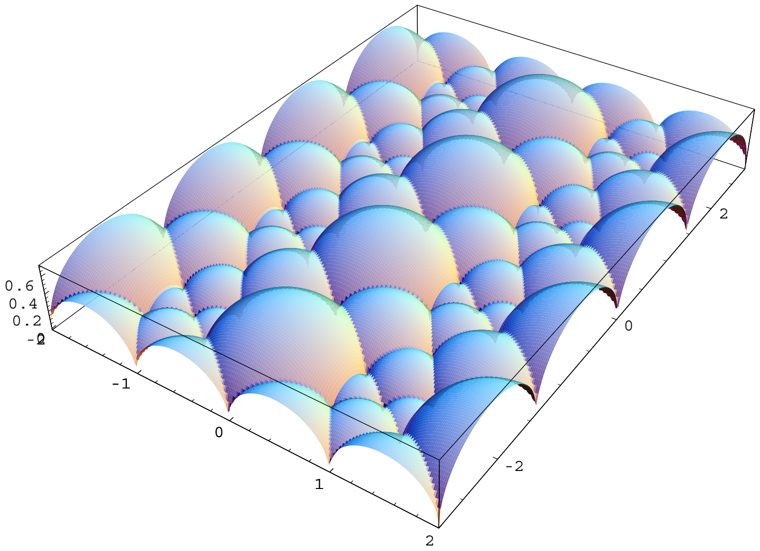

Figure 3: Fundamental Polyhedron of Γ

Figure 2 represents the bases of the 64 spheres mentioned. Notice that these bases intersects in groups of 3. Let P be the set of these intersections. Each side of D is included in one of the mentioned half spheres or in one of the four half planes $x = \pm 2$ and $y = \pm 2\sqrt{2}$. Figure 3 represents D. The intersections of 2 of the sides of D are geodesic lines of $\mathbb{H}^3$ with extremes in elements of P. Now it is easy to see that D is the convex hull of the intersections of pairs of sides of D. Abusing of the notation we can refer to D as the convex hull of P in $\mathbb{H}^3$. A tedious by straightforward computation shows that each element of P is of one of the forms of (3). By Lemma 3 every element of P is in the exterior part of any isometric sphere of $\Gamma \setminus \Gamma_\infty$. Since D_1 is convex, D_1 contains the convex hull of D, that is $D_1 \supseteq D$. ∎

Now that the fundamental polyhedron of Γ has been obtained in Proposition 4, the next step is looking for a set of generators of Γ. It is well known that the side-pairing transformations of the sides of D generate Γ. A side-pairing transformation of D is an element $g \in \Gamma$ such that $s_g = D \cap g^{-1}(D)$ has dimension 2. The side-pairing transformations of Γ_∞ are

$$S = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 4\sqrt{-2} \\ 0 & 1 \end{pmatrix} \tag{8}$$

and their inverses and they are also side-pairing transformations of Γ. A straightforward computation shows that the elements of (4), (5), (6) and (7) and their inverses are also side-pairing transformations. To show that there are not more side-pairing transformations it is enough to realize that the sides s_g of the set of pairing transformations obtained cover the boundary of D.

The sets s_g for g a side-pairing transformation are call sides of D. It is clear that $s'_g = g(s_g) = s_{g^{-1}}$. The association $s \mapsto s'$ is call the pairing of D. Figure 4 represents the projection of the fundamental polyhedron D in $\mathbb{C}$, the sided-pairing transformations associated to each side and the pairing. Its important to notice that the generators S,T and the corresponding inverses are associated to the vertical faces of D.

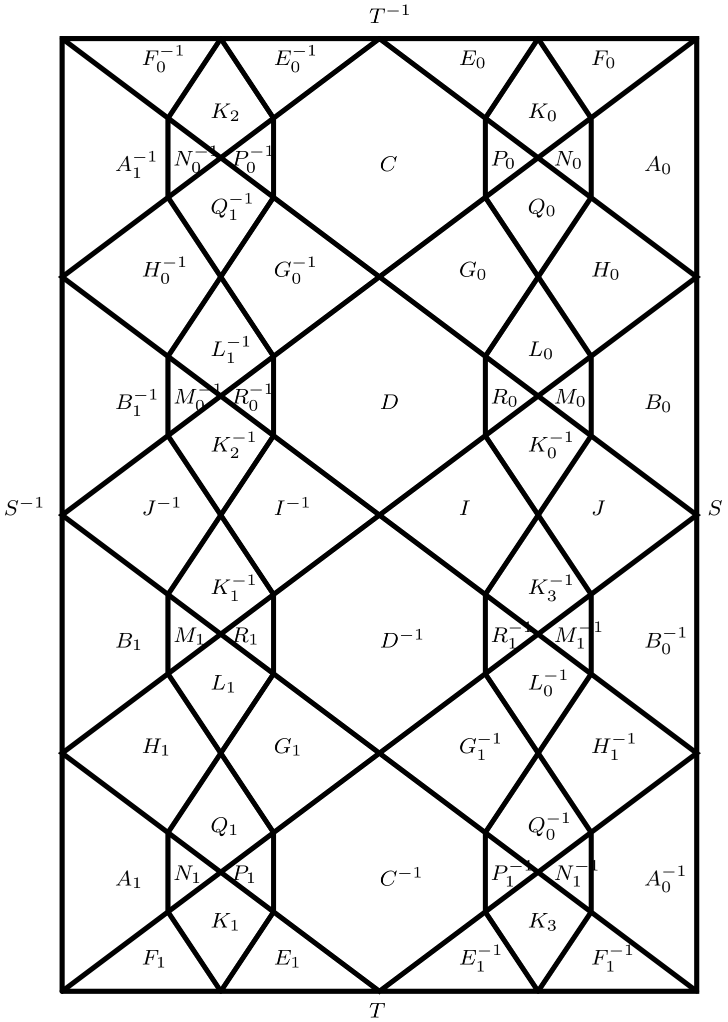

Figure 4: Sided-Pairing Transformations.

By Poincaré‘s Method, Γ has a presentation where the generators are the side-pairing transformation and the relations are the reflection and cycle relations. The reflection relations are of the form $g^2 = 1$ where g is a side-pairing transformation such that $s'_g = s_g$. The domain D does not yield to any reflection relation as it is clear in Figure 4. The cycle relations are associated to cycles of D which are sequences of the form $(a_1, s_1, a_2, s_2, ..., a_n, s_n)$ such that:

1. $s_1, ..., s_n$ are sides of D with $s_{i+1} \neq s'_i$ for all $i = 1, ..., n-1$.

2. $a_1, ..., a_n$ are edges of D with $s_i' \cap s_{i+1} = a_{i+1}$ for all $i = 1, ..., n-1$, in fact, s_i' and s_{i+1} are the only two sides containing the edge a_i.
3. $a_{n+1} = a_1$ and $g_i(a_i) = a_{i+1}$ for all $i = 1, ..., n$, where $g_i \in \Gamma$ is the side-pairing transformation associated to s_i.

If $(a_1, s_1, ..., a_n, s_n)$ is a cycle of D, then there exist $m \in \mathbb{Z}^+$ such that $(g_n g_{n-1} \dots g_2 g_1)^m = 1$. These are the cycle relations.

Looking for the cycles of D is now another tedious straightforward calculation. Figure 5 represents the cycles of edges of the fundamental polyhedron D. Each cycle give rise to a cycle relation which is one of the relations given in Theorem 1.

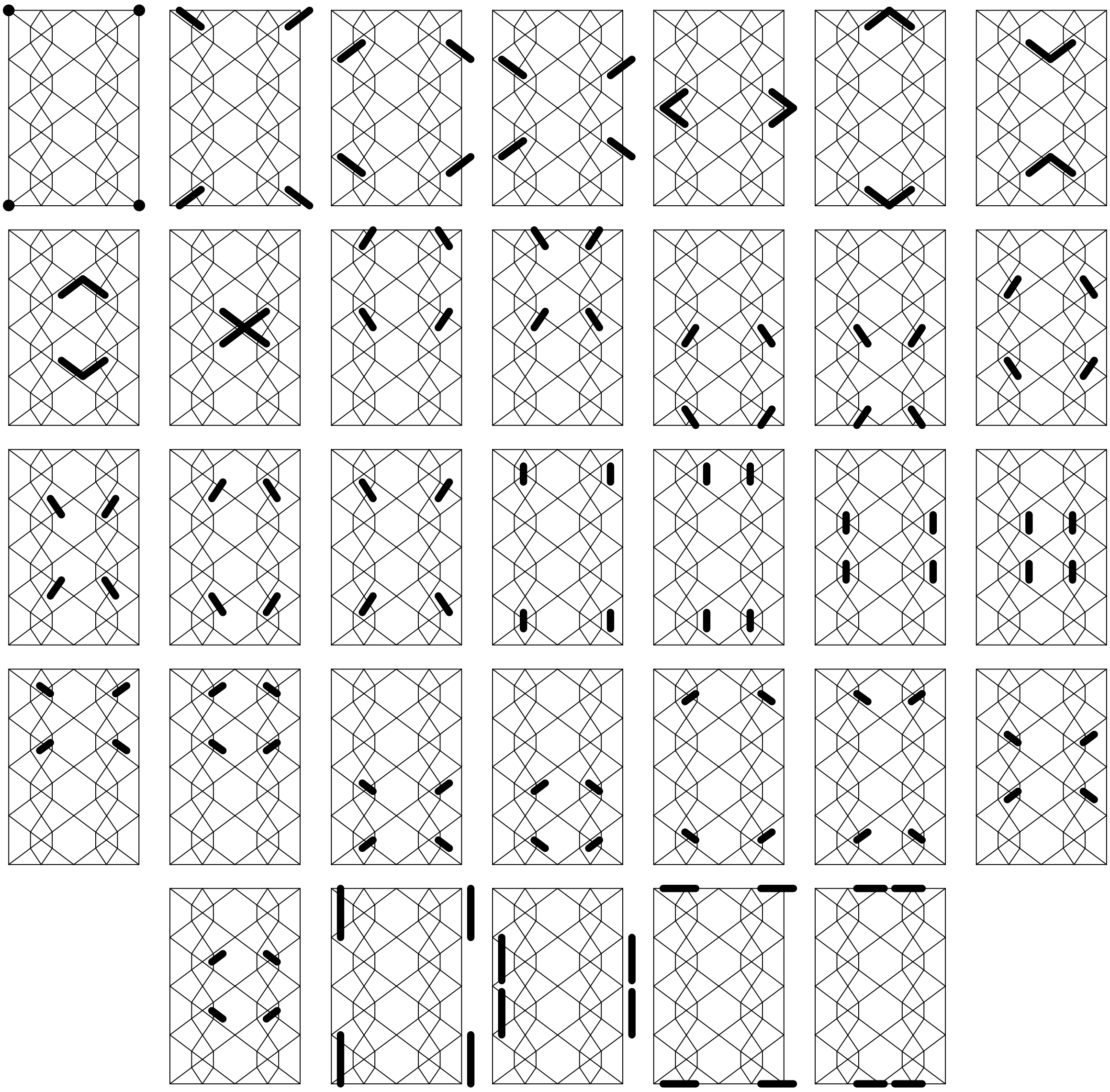

Figure 5: Cycles.

References

[1] **A. F. Beardon,** *The geometry of Discrete Groups*, **Springer, 1983.**

[2] **J. Elstrodt, F.Grunewald y J. Mennicke,** *Groups Acting on Hyperbolic Space. Harmonic Analysis and Number Theory*, **Springer, 1998.**

[3] **B. Fine,** *The Algebraic structure of the Bianchi Groups*, **Marcel Dekker, 1989.**

[4] **G. Higman,** *Units in group rings*, **D. Phil. Thesis, University of Oxford, Oxford, 1940.**

[5] **E. Jespers, M. M. Parmenter,** *Units of group rings of groups of order 16*, **Glasgow Math. J. 35, 367-379, 1993.**

[6] **E. Jespers y Á. del Río,** *A structure theorem for the unit group of the integral group ring of some finite groups*, **J. Reine Angew. Math. 521, 99-117, 2000.**

[7] **A. Pita, Á. del Río y M. Ruiz,** *Group of units of integral group rings of kleinian type*, **(to appear in Trans. AMS).**

[8] **Á. del Río y M. Ruiz,** *Computing large direct products of free groups in integral group rings*, **Comm. Algebra 30 (4), 1751-1767, 2002.**

[9] **S.K. Sehgal,** *Units in Integral Group rings*, **Longman Scientific and Technical Essex, 1993.**

Antonio Pita
Dep. Matemáticas
Universidad de Murcia
Campus de Espinardo
30100 Murcia, Spain
antopita@um.es