ZASSENHAUS CONJECTURE (ZC1) ON TORSION UNITS OF INTEGRAL GROUP RINGS FOR SOME METABELIAN GROUPS

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Abstract. We prove a conjecture of Zassenhaus that every normalized torsion unit of the integral group ring $\mathbb{Z}G$ of a finite group G is rationally conjugate to a group element for some metabelian groups including metacyclic groups G containing a normal cyclic group A such that G/A is cyclic of prime power order. The relative prime case was done in [11].

1. Introduction

A long standing Conjecture of Zassenhaus states:

$$\text{(ZC1)}\quad \begin{array}{l}\text{If } G \text{ is a finite group and } u \text{ is a torsion element of } U_1(\mathbb{Z}G)\\ \text{then there is a unit } v \text{ of } \mathbb{Q}G \text{ such that } v^{-1}uv \in G.\end{array}$$

This conjecture has been verified for some families of finite groups including nilpotent groups [14] and some split extensions $G = A \rtimes X$ with $|A|$ and $|X|$ relatively prime (see [10, 11] for A and X cyclic, the generalization of [5] for A cyclic and X abelian and the recent result of [4] for A a p-group and X abelian). Also (ZC1) is known for $G = \langle a \rangle \rtimes \langle x \rangle$ where $|x|$ is a prime number (see [6] and [7]).

The main result of this paper is the following theorem, where $\pi(G)$ denotes the set of prime divisors of the order of a group G and if G is nilpotent then G_p denotes the Sylow p-subgroup of G and $G_{p'}$ the product of Sylow q-subgroups of G for $q \in \pi(G) \setminus \{p\}$.

Theorem 1. *Let G be a finite group and assume that $G = AX$ for A a cyclic normal subgroup and X an abelian subgroup of G. Assume additionally that $\pi(A) \cap \pi(X)$ has at most one element and if such an element, say p, exists then A_p and $X_{p'}$ commute elementwise. Then every torsion unit of $U_1(\mathbb{Z}G)$ is conjugate in $\mathbb{Q}G^*$ to an element of G.*

As special cases of Theorem 1 we get the following corollaries.

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Corollary 2. *Let* G *be a finite group and assume that* $G = AX$ *for* A *a cyclic normal subgroup of* G *and* X *an abelian subgroup of* G*. If* $\pi(A) \cap \pi(X) = \{p\}$ *and* $(p-1, |X|) = 1$ *then every torsion unit of* $U_1(\mathbb{Z}G)$ *is conjugate in* $\mathbb{Q}G^*$ *to an element of* G*.*

Corollary 3. *Let* G *be a finite group and assume that* $G = AX$ *for* A *a cyclic normal subgroup of* G *and* X *an abelian* p*-subgroup of* G*. Then every torsion unit of* $U_1(\mathbb{Z}G)$ *is conjugate in* $\mathbb{Q}G^*$ *to an element of* G*.*

Corollary 4. *If* G *is metacyclic with a normal cyclic subgroup* A *such that* G/A *is cyclic and* $|A|$ *and* $[G : A]$ *have at most one common prime divisor* p *and* $(p-1, [G : A]) = 1$ *then every torsion unit of* $U_1(\mathbb{Z}G)$ *is conjugate in* $\mathbb{Q}G^*$ *to an element of* G*.*

Corollary 5. *If* G *has a cyclic normal subgroup* A *such that* G/A *is a cyclic* p*-group then every torsion unit of* $U_1(\mathbb{Z}G)$ *is conjugate in* $\mathbb{Q}G^*$ *to an element of* G*.*

Notice that Theorem 1 extends the results mentioned from [10, 11, 7, 5] and [6] but not the one from [4]. However our proof is independent of the results extended and therefore provides shorter proofs. The pay off is by using a theorem of Cliff and Weiss [2] (see Theorem 7 below). This approach follows the matrix-like program suggested in [8] (see also [13, page 249] which leads to the following problem, which for $n = 1$ is just (ZC1).

Problem 6. *Let* $\mathrm{SGL}_n(\mathbb{Z}G)$ *be the kernel of the group homomorphism* $\mathrm{GL}_n(\mathbb{Z}G) \rightarrow \mathrm{GL}_n(\mathbb{Z})$ *given by applying the augmentation homomorphism componentwise. Is every torsion unit of* $\mathrm{SGL}_n(\mathbb{Z}G)$ *conjugate in* $\mathrm{GL}_n(\mathbb{Q}G)$ *to a diagonal matrix of elements of* G*?*

The mentioned theorem of Cliff and Weiss is their solution of Problem 6 for nilpotent groups and matrices of arbitrary size.

Theorem 7. (Cliff-Weiss) *If* N *is nilpotent then Problem 6 has a positive solution for every positive integer* n *if and only if the number of non cyclic Sylow subgroups of* N *is at most 1.*

2. Some Lemmas

We use the following notation for G a finite group, R a ring, $x, y \in G$, $C \subseteq G$, $u \in RG$ and A a normal subgroup of G:

$$\begin{array}{rcl}
o(x) & = & \text{Order of } x \\
x^y & = & y^{-1}xy \\
(x,y) & = & xyx^{-1}y^{-1} \\
x^G & = & \{x^g : g \in G\} \\
\mathcal{C}l(G) & = & \text{Set of conjugacy classes of } G \\
\pi(x) & = & \pi(\langle x \rangle) \\
u_x & = & \text{Coefficient of } x \text{ in } u \\
u_C & = & \sum_{c \in C} u_c \\
R^* & = & \text{Group of units of } R = U(R) \\
U_1(RG) & = & \{u \in U(RG) : u_G = 1\} = \text{Units of augmentation one in RG} \\
\Delta_R(G,A) & = & \text{Kernel of the natural projection: } RG \to R(G/A)
\end{array}$$

Notice that if $g \in G$ then u_{g^C} is denoted $\tilde{u}(g)$ in [13]. If N is a finite nilpotent group, $p \in \pi(N)$ and $\pi \subseteq \pi(G)$ let N_p denote the Sylow p-subgroup of N, $N_\pi = \prod_{q \in \pi} N_q$, $N_{\pi'} = N_{\pi(G) \setminus \pi}$ and $N_{p'} = N_{\pi(G) \setminus \{p\}}$. Moreover if g is an element of a group then g_π denotes the π-part of g, that is the projection of x in the decomposition $\langle x \rangle = \langle x \rangle_\pi \times \langle x \rangle_{\pi'}$.We recall the following useful criterion for (ZC1) to hold. Notice that the equivalence between 1 and 2 is a trivial consequence of [13, Lemma 41.5] and the equivalence of 2 and 3 is obvious because $1 = v_G = \sum_{C \in \mathcal{C}l(G)} v_C$.

Lemma 1. *The following are equivalent for a torsion element u of $U_1(\mathbb{Z}G)$:*

1. *u is conjugate in $\mathbb{Q}G^*$ to an element of G.*
2. *For every $v \in \langle u \rangle$ the cardinality of $\{C \in \mathcal{C}l(G) : v_C \neq 0\}$ is 1.*
3. *For every $v \in \langle u \rangle$ and every $C \in \mathcal{C}l(G)$ one has $v_C \geq 0$.*

We start proving some easy and well known lemmas.

Lemma 2. *Let A be an abelian normal subgroup of G and $g \in G$ such that $g^G \subseteq Ag$. Then $\langle (A,g) \rangle g^G = g^G$.*

Proof. The inclusion $g^G \subseteq \langle (A,g) \rangle g^G$ is clear. If $x \in \langle (A,g) \rangle$ then $x = (a_1, g) \cdots (a_k, g)$ for some $a_1, \ldots, a_k \in A$. Therefore to show $xg^G \subseteq g^G$ one may assume without loss of generality that $k = 1$, that is $x = (a, g)$ for some $a \in A$. Let $z \in g^G$ and $b = zg^{-1}$. By hypothesis $b \in g^G g^{-1} \subseteq A$ and hence $b_1 = b^g \in A$. Then

$$xz = (a, g)z = aga^{-1}g^{-1}bg = aga^{-1}b_1 = agb_1a^{-1} = abga^{-1} = aza^{-1} \in z^G = g^G.$$

□

Lemma 3. *Let A be a normal cyclic subgroup of G and X a subgroup of G. If $g \in A_pX$ and $p \in \pi(g) \setminus \pi(X)$ then $(g, A_p) = 1$.*

Proof. Write $g = ax$ with $a \in A_p$ and $x \in X$ and let $A_p = \langle b \rangle$, $p^m = o(b)$ and $s = o(x)$. Set $b^x = b^j$ for some integer j. Let n be the order of j in $\mathbb{Z}^*_{p^m}$. Then $b = b^{x^s} = b^{j^s}$ and thus $n|s$.

By means of contradiction assume that $(g, A_p) \neq 1$ or equivalently $n \neq 1$. We observe that $j \not\equiv 1 \mod p$ (otherwise $j^{p^{m-1}} \equiv 1 \mod p^m$). Now from $j^s \equiv 1 \mod p^m$ and $j \not\equiv 1 \mod p$ we deduce that

$$1 + j + j^2 + \ldots + j^{s-1} \equiv 0 \mod p^m$$

and hence

$$g^s = (ax)^s = a^{1+j+j^2+\ldots+j^{s-1}}x^s = 1.$$

It follows that $p|o(g)|s$, a contradiction. □

For every integer n and ring R let $\mathrm{tr} : M_n(R) \to R/[R, R]$ be the map which associates to a matrix $a \in M_n(R)$ the sum of its diagonal entries modulo $[R, R]$. Notice that if R is a commutative ring and G is a group then $[RG, RG]$ is the R-span of the elements of the form $[g, h]$ for $g, h \in G$ and also $[RG, RG] = \{u \in RG : u_C = 0 \text{ for all } C \in \mathcal{C}l(G)\}$ (see [13, 41.1]). This implies that

$$u \equiv \sum_{t \in T} u_{t^G}\, t \mod [RG, RG] \tag{1}$$

where T is a set of representatives of the conjugacy classes of G and $RG/[RG, RG]$ is isomorphic as an R-module to $\oplus_{t \in T} Rt$ and therefore it is also isomorphic as an R-module to $Z(RG)$. In particular, if the characteristic of R is 0, then $RG/[RG, RG]$ is a torsion-free additive group.Let now N be a normal subgroup of G and consider the (RG, RN)-bimodule $M = RG \otimes_{RN} RN$. If B is a set of representatives of cosets of N in G then

$\{b\otimes 1 : b\in B\}$ is a basis of M_{RN}. The matrix representation in this basis of the ring embedding $RG\to \mathrm{End}(M_{RN})$ that associates to every element a of RG to the left multiplication map $M\ni x\to ax$ induces a ring map $\rho : RG\to M_n(RN)$ where $n=[G:N]$. We will use the following formula, for $u\in RG$ and S a set of representatives of the conjugacy classes of N, which is proved(for the split case but the argument holds generally) in [13, 41.10]:

$$\mathrm{tr}\rho(u)=\sum_{s\in S} u_{s^G}[\mathrm{Cen}_G(s):\mathrm{Cen}_N(s)]s. \tag{2}$$

We are ready to prove Theorem 1.

3. Proof of Theorem 1

By means of contradiction we assume that G is a finite group of minimal order among those satisfying the conditions of Theorem 1 such that $U_1(\mathbb{Z}G)$ contains a torsion element u which is not conjugate in $\mathbb{Q}G^*$ to an element of G. Assume that u has minimal order among the elements satisfying this property.Applying Lemma 1 with the hypothesis that u is not conjugate in $\mathbb{Q}G$ to an element of G one deduces that there is an element $v\in\langle u\rangle$ such that $v_C<0$ for some $C\in\mathcal{C}l(G)$. Applying again Lemma 1 to v one deduces that v is not conjugate in $\mathbb{Q}G^*$ to an element of G and by the minimality of the order of u one can take $u=v$, that is $u_C<0$ for some $C\in\mathcal{C}l(G)$. This will be in contradiction with:

Main Claim: $u_C\ge 0$ for every $C\in\mathcal{C}l(G)$.Since A is abelian, $\mathbb{Z}G^*\cap(1+\Delta(G)\Delta(A))$ is torsion free (see [1] or [13, 31.3] under the additional hypothesis that G/A is abelian which holds here and [9] without this hypothesis). Using this it is easy to show that if v and w are two torsion units of $U_1(\mathbb{Z}G)$ such that $v\equiv w \mod \Delta(G)\Delta(A)$ then $o(v)=o(w)$. We will use this several times without specific mention.

If B is a subgroup of A then B is normal in G and it is easy to see that G/B also satisfies the conditions of Theorem 1. Therefore, if B is non trivial then the minimality of the order of G implies that (ZC1) holds for G/B. We will also use this several times without specific mention.Let $\pi=\pi(G)\setminus\pi(X)$ and $\lambda=\pi(G)\setminus\pi(A)$ so that $\pi(G)$ is the disjoint union of π, λ and possibly $\{p\}$.

Set $\Delta(G) = \Delta_{\mathbb{Z}}(G,G)$ and $\Delta(A) = \Delta_{\mathbb{Z}}(A,A)$. Using that G/A is abelian and hence every torsion unit of augmentation one in $\mathbb{Z}(G/A)$ is an element of G/A and the Whitcomb argument [13, 30.5] one deduces that there is $y \in G$ such that $u \equiv y \mod \Delta(G)\Delta(A)$. Then $u_\tau \equiv y_\tau \mod \Delta(G)\Delta(A)$ for every $\tau \subseteq \pi(G)$. In particular $y_q \in A_q$ if $q \in \pi$. Moreover if $\lambda_1 = \{p\} \cup \lambda$ and $z_0 = y_{\lambda_1}$ then $\pi(z_0) \subseteq \lambda_1$. Since A_pX is a Hall λ_1 subgroup of G then z_0 is conjugate to an element of A_pX [12, 9.1.3]. Conjugating u by this element we assume that $z_0 \in A_pX = (A_pX_p) \times X_\lambda$ and therefore $y_p \in A_pX_p$ and $y_\lambda \in X_\lambda = X_{p'}$.*Claim 1.* $\pi(A) \subseteq \pi(u)$.

This is basically a consequence of [3, 2.2]. We include an elementary (slightly different) proof for the sake of completeness. By means of contradiction let $q \in \pi(A) \setminus \pi(u)$. Let $\bar{G} = G/A_q$ and use the bar notation for reduction modulo A_q. Since $A_q \neq 1$, (ZC1) holds for $\bar{G} = G/A_q$, so that $\bar{u}_D \geq 0$ for every $D \in \mathcal{C}l(\bar{G})$. However we know that $u_C < 0$ for some $C \in \mathcal{C}l(G)$. Let $g \in C$. By [13, 38.11], if $h \in G$ is an element of C whose order is multiple of q then $u_{h^G} = 0$. This shows that $g_q = 1$. However we are going to show that if $h_q = 1$ and $\bar{h} \sim \bar{g}$ then $h \sim g$, so that $0 \geq \bar{u}_{\bar{C}} = u_C < 0$, a contradiction. Indeed, if $\bar{h} \sim \bar{g}$ then $h \sim ag$, $a \in A_q$. Since $o(h)$ is not multiple of q, one has that $o(h) = o(g)$ and therefore $(a,g) \neq 1$. Write $a^g = a^j$ and $q^m = o(a)$. As in the proof of Lemma 3 we have that $j \not\equiv 1 \mod q$. Let $k(j-1) \equiv 1 \mod q^m$. Then $a^k g a^{-k} = a^{k(j-1)} g = ag \sim h$, and this finishes the proof of Claim 1.*Claim 2.* $(y_t, A_\pi) = 1$ for every $t \in \pi(G)$.

The statement is obvious if $t \in \pi$ because $y_t \in A_\pi$ and A_π is abelian. Otherwise $y_t \in \langle y_{\pi(X)} \rangle$ and therefore it is enough to show that $(y_{\pi(X)}, A_\pi) = 1$. On the other hand $A_\pi = \prod_{q\in\pi} A_q$ and thus to prove Claim 2 one only needs to show that $(y_{\pi(X)}, A_q) = 1$ for every $q \in \pi$. Let $q \in \pi$. Then $u_q \neq 1$, by Claim 1. On the other hand, $u_{\pi(X)} u_q \equiv y_{\pi(X)} y_q \mod \Delta(G)\Delta(A)$ and hence $q | o(u_{\pi(X)}) o(u_q) = o(u_{\pi(X)} u_q) = o(y_{\pi(X)} y_q)$. Since $y_q \in A_q$, applying Lemma 3 we deduce that $(y_{\pi(X)} y_q, A_q) = 1$, or equivalently $(y_{\pi(X)}, A_q) = 1$. This proves Claim 2.*Claim 3.* $\pi(A) = \pi(u)$ and $u \equiv 1 \mod \Delta(G, M)$, where $M = \mathrm{Cen}_{AX_p}(A_\pi)$.By Claim 2, $z = y_\lambda \in \mathrm{Cen}_{X_{p'}}(A_\pi)$. Recall that we have been careful to pick $z_0 \in A_pX = (A_pX_p) \times X_\lambda$ and therefore $z \in X_\lambda$. Thus z commutes with A_p,A_π and X and therefore is a central element of G. Furthermore $uz^{-1} \equiv y_{\pi(A)} \mod \Delta(G)\Delta(A)$ and therefore $o(uz^{-1}) = o(y_{\pi(A)})$. Since

z is central, uz^{-1} is not conjugate in $\mathbb{Q}G^*$ to an element of G. Now the minimality of $o(u)$ implies that $o(y) = o(u) = o(uz^{-1}) = o(y_{\pi(A)}) = o(y)/o(z)$ and so $z = 1$. This shows that $y = y_\pi y_p \in A_\pi A_p X_p = AX_p$. Combining this with Claim 2 one deduces that $y \in M$ and hence $u \equiv y \equiv 1 \mod \Delta(G, M)$. This establishes Claim 3.

Finally we address the proof of the Main Claim. Let $g \in C \in \mathcal{C}l(G)$.Assume first $g \notin \mathrm{Cen}_G(A)$. Then $B = \langle (a, g) \rangle \neq 1$ for some $a \in A$. Hence (ZC1) holds for $\bar{G} = G/B$. Using Lemma 1 there is $D \in \mathcal{C}l(\bar{G})$ such that $\bar{u}_E = 0$ for every $D \neq E \in \mathcal{C}l(\bar{G})$. Then $\bar{u}_{\bar{D}} = 1$, since $\bar{u}_{\bar{G}} = u_G = 1$, On the other hand $C = g^G \subseteq G'g \subseteq Ag$. Using Lemma 2 one has that $BC = C$ and this implies that $u_C = \bar{u}_{\bar{C}} \geq 0$.

Now let $g \in \mathrm{Cen}_G(A)$ and let $N = \langle M, g \rangle$. Let $g = hx$ with $h \in \langle g \rangle_{\pi(A)}$ and $x \in \langle g \rangle_\lambda$. Then $h \in M$ and so $N = \langle M, x \rangle$ and $x^{h_0} \in X$ for some $h_0 \in G$. Since $x \in \mathrm{Cen}_G(A)$ then $x^{h_0} \in \mathrm{Cen}_X(A) \subseteq Z(G)$. Therefore $x \in Z(G)$ and we conclude that $N = \langle M, x \rangle = A_\pi \times \mathrm{Cen}_{A_p X_p}(A_\pi) \times \langle x \rangle$, a nilpotent group. Furthermore N_q is cyclic for every $p \neq q \in \pi(N)$.Let $n = [G : N]$ and let $\rho : \mathbb{Q}G \to M_n(\mathbb{Q}N)$ as explained at the end of section 2. Notice that ρ restricts to a group homomorphism $\mathbb{Z}G^* \cap (1 + \Delta(G, N)) \to \mathrm{SGL}_n(\mathbb{Z}N)$. By Claim 3, $u \in \mathbb{Z}G^* \cap (1 + \Delta(G, M)) \subseteq \mathbb{Z}G^* \cap (1 + \Delta(G, N))$ and hence $\rho(u)$ is a torsion unit of $\mathrm{SGL}_n(\mathbb{Z}N)$. In the previous paragraph we have checked that N satisfies the hypothesis of Theorem 7. Thus there is $\alpha \in \mathrm{GL}_n(\mathbb{Q}N)$ such that $\rho(u) = \alpha^{-1} d\alpha$ for d a diagonal matrix formed by elements of N. This implies that $\mathrm{tr}\rho(u) = \mathrm{tr}d$ is of the form $\sum_{b \in N} k_b b$ with k_b non negative integers. By using (1) and (2) one has that $u_D \geq 0$ for every $D \in \mathcal{C}l(G)$ such that $D \subseteq N$, and in particular $u_C \geq 0$. This completes the proof of the Main Claim and of Theorem 1. □

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