ZASSENHAUS CONJECTURE (ZC1) ON TORSION UNITS OF INTEGRAL GROUP RINGS FOR SOME METABELIAN GROUPS

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ABSTRACT. We prove a conjecture of Zassenhaus that every normalized torsion unit of the integral group ring $\mathbb{Z}G$ of a finite group G is rationally conjugate to a group element for some metabelian groups including metacyclic groups G containing a normal cyclic group A such that G/A is cyclic of prime power order. The relative prime case was done in [11].

1. INTRODUCTION

A long standing Conjecture of Zassenhaus states:

(ZC1) If G is a finite group and u is a torsion element of $U_1(\mathbb{Z}G)$ then there is a unit v of $\mathbb{Q}G$ such that $v^{-1}uv \in G$.

This conjecture has been verified for some families of finite groups including nilpotent groups [14] and some split extensions $G = A \rtimes X$ with $|A|$ and $|X|$ relatively prime (see [10, 11] for A and X cyclic, the generalization of [5] for A cyclic and X abelian and the recent result of [4] for A a p -group and X abelian). Also (ZC1) is known for $G = \langle a \rangle \rtimes \langle x \rangle$ where $|x|$ is a prime number (see [6] and [7]).

The main result of this paper is the following theorem, where $\pi(G)$ denotes the set of prime divisors of the order of a group G and if G is nilpotent then G_p denotes the Sylow p -subgroup of G and $G_{p'}$ the product of Sylow q -subgroups of G for $q \in \pi(G) \setminus \{p\}$.

Theorem 1. *Let G be a finite group and assume that $G = AX$ for A a cyclic normal subgroup and X an abelian subgroup of G . Assume additionally that $\pi(A) \cap \pi(X)$ has at most one element and if such an element, say p , exists then A_p and $X_{p'}$ commute elementwise. Then every torsion unit of $U_1(\mathbb{Z}G)$ is conjugate in $\mathbb{Q}G^*$ to an element of G .*

As special cases of Theorem 1 we get the following corollaries.

Research supported by D.G.I. of Spain, Fundación Séneca of Murcia and NSERC of Canada.
AMS subject classification: Primary 16U60, Secondary 11R27, 16A26.

Corollary 2. *Let G be a finite group and assume that $G = AX$ for A a cyclic normal subgroup of G and X an abelian subgroup of G . If $\pi(A) \cap \pi(X) = \{p\}$ and $(p-1, |X|) = 1$ then every torsion unit of $U_1(\mathbb{Z}G)$ is conjugate in $\mathbb{Q}G^*$ to an element of G .*

Corollary 3. *Let G be a finite group and assume that $G = AX$ for A a cyclic normal subgroup of G and X an abelian p -subgroup of G . Then every torsion unit of $U_1(\mathbb{Z}G)$ is conjugate in $\mathbb{Q}G^*$ to an element of G .*

Corollary 4. *If G is metacyclic with a normal cyclic subgroup A such that G/A is cyclic and $|A|$ and $[G : A]$ have at most one common prime divisor p and $(p-1, [G : A]) = 1$ then every torsion unit of $U_1(\mathbb{Z}G)$ is conjugate in $\mathbb{Q}G^*$ to an element of G .*

Corollary 5. *If G has a cyclic normal subgroup A such that G/A is a cyclic p -group then every torsion unit of $U_1(\mathbb{Z}G)$ is conjugate in $\mathbb{Q}G^*$ to an element of G .*

Notice that Theorem 1 extends the results mentioned from [10, 11, 7, 5] and [6] but not the one from [4]. However our proof is independent of the results extended and therefore provides shorter proofs. The pay off is by using a theorem of Cliff and Weiss [2] (see Theorem 7 below). This approach follows the matrix-like program suggested in [8] (see also [13, page 249] which leads to the following problem, which for $n = 1$ is just (ZC1).

Problem 6. *Let $\text{SGL}_n(\mathbb{Z}G)$ be the kernel of the group homomorphism $\text{GL}_n(\mathbb{Z}G) \rightarrow \text{GL}_n(\mathbb{Z})$ given by applying the augmentation homomorphism componentwise. Is every torsion unit of $\text{SGL}_n(\mathbb{Z}G)$ conjugate in $\text{GL}_n(\mathbb{Q}G)$ to a diagonal matrix of elements of G ?*

The mentioned theorem of Cliff and Weiss is their solution of Problem 6 for nilpotent groups and matrices of arbitrary size.

Theorem 7. (Cliff-Weiss) *If N is nilpotent then Problem 6 has a positive solution for every positive integer n if and only if the number of non cyclic Sylow subgroups of N is at most 1.*

2. SOME LEMMAS

We use the following notation for G a finite group, R a ring, $x, y \in G$, $C \subseteq G$, $u \in RG$ and A a normal subgroup of G :

$$\begin{aligned}
 o(x) &= \text{Order of } x \\
 x^y &= y^{-1}xy \\
 (x, y) &= xyx^{-1}y^{-1} \\
 x^G &= \{x^g : g \in G\} \\
 Cl(G) &= \text{Set of conjugacy classes of } G \\
 \pi(x) &= \pi(\langle x \rangle) \\
 u_x &= \text{Coefficient of } x \text{ in } u \\
 u_C &= \sum_{c \in C} u_c \\
 R^* &= \text{Group of units of } R = U(R) \\
 U_1(RG) &= \{u \in U(RG) : u_G = 1\} = \text{Units of augmentation one in } RG \\
 \Delta_R(G, A) &= \text{Kernel of the natural projection: } RG \rightarrow R(G/A)
 \end{aligned}$$

Notice that if $g \in G$ then u_{gC} is denoted $\tilde{u}(g)$ in [13]. If N is a finite nilpotent group, $p \in \pi(N)$ and $\pi \subseteq \pi(G)$ let N_p denote the Sylow p -subgroup of N , $N_\pi = \prod_{q \in \pi} N_q$, $N_{\pi'} = N_{\pi(G) \setminus \pi}$ and $N_{p'} = N_{\pi(G) \setminus \{p\}}$. Moreover if g is an element of a group then g_π denotes the π -part of g , that is the projection of x in the decomposition $\langle x \rangle = \langle x \rangle_\pi \times \langle x \rangle_{\pi'}$. We recall the following useful criterion for (ZC1) to hold. Notice that the equivalence between 1 and 2 is a trivial consequence of [13, Lemma 41.5] and the equivalence of 2 and 3 is obvious because $1 = v_G = \sum_{C \in Cl(G)} v_C$.

Lemma 1. *The following are equivalent for a torsion element u of $U_1(\mathbb{Z}G)$:*

- (1) u is conjugate in $\mathbb{Q}G^*$ to an element of G .
- (2) For every $v \in \langle u \rangle$ the cardinality of $\{C \in Cl(G) : v_C \neq 0\}$ is 1.
- (3) For every $v \in \langle u \rangle$ and every $C \in Cl(G)$ one has $v_C \geq 0$.

We start proving some easy and well known lemmas.

Lemma 2. *Let A be an abelian normal subgroup of G and $g \in G$ such that $g^G \subseteq Ag$. Then $\langle (A, g) \rangle g^G = g^G$.*

Proof. The inclusion $g^G \subseteq \langle (A, g) \rangle g^G$ is clear. If $x \in \langle (A, g) \rangle$ then $x = (a_1, g) \cdots (a_k, g)$ for some $a_1, \dots, a_k \in A$. Therefore to show $xg^G \subseteq g^G$ one may assume without loss of generality

that $k = 1$, that is $x = (a, g)$ for some $a \in A$. Let $z \in g^G$ and $b = zg^{-1}$. By hypothesis $b \in g^G g^{-1} \subseteq A$ and hence $b_1 = b^g \in A$. Then

$$xz = (a, g)z = aga^{-1}g^{-1}bg = aga^{-1}b_1 = agb_1a^{-1} = abga^{-1} = aza^{-1} \in z^G = g^G.$$

□

Lemma 3. *Let A be a normal cyclic subgroup of G and X a subgroup of G . If $g \in A_p X$ and $p \in \pi(g) \setminus \pi(X)$ then $(g, A_p) = 1$.*

Proof. Write $g = ax$ with $a \in A_p$ and $x \in X$ and let $A_p = \langle b \rangle$, $p^m = o(b)$ and $s = o(x)$. Set $b^x = b^j$ for some integer j . Let n be the order of j in $\mathbb{Z}_{p^m}^*$. Then $b = b^{x^s} = b^{j^s}$ and thus $n|s$.

By means of contradiction assume that $(g, A_p) \neq 1$ or equivalently $n \neq 1$. We observe that $j \not\equiv 1 \pmod{p}$ (otherwise $j^{p^{m-1}} \equiv 1 \pmod{p^m}$). Now from $j^s \equiv 1 \pmod{p^m}$ and $j \not\equiv 1 \pmod{p}$ we deduce that

$$1 + j + j^2 + \dots + j^{s-1} \equiv 0 \pmod{p^m}$$

and hence

$$g^s = (ax)^s = a^{1+j+j^2+\dots+j^{s-1}} x^s = 1.$$

It follows that $p|o(g)|s$, a contradiction. □

For every integer n and ring R let $\text{tr} : M_n(R) \rightarrow R/[R, R]$ be the map which associates to a matrix $a \in M_n(R)$ the sum of its diagonal entries modulo $[R, R]$. Notice that if R is a commutative ring and G is a group then $[RG, RG]$ is the R -span of the elements of the form $[g, h]$ for $g, h \in G$ and also $[RG, RG] = \{u \in RG : u_C = 0 \text{ for all } C \in \mathcal{Cl}(G)\}$ (see [13, 41.1]). This implies that

$$(1) \quad u \equiv \sum_{t \in T} u_t g \, t \pmod{[RG, RG]}$$

where T is a set of representatives of the conjugacy classes of G and $RG/[RG, RG]$ is isomorphic as an R -module to $\bigoplus_{t \in T} R t$ and therefore it is also isomorphic as an R -module to $Z(RG)$. In particular, if the characteristic of R is 0, then $RG/[RG, RG]$ is a torsion-free additive group. Let now N be a normal subgroup of G and consider the (RG, RN) -bimodule $M = RG \otimes_{RN} RN$. If B is a set of representatives of cosets of N in G then

$\{b \otimes 1 : b \in B\}$ is a basis of M_{RN} . The matrix representation in this basis of the ring embedding $RG \rightarrow \text{End}(M_{RN})$ that associates to every element a of RG to the left multiplication map $M \ni x \rightarrow ax$ induces a ring map $\rho : RG \rightarrow M_n(RN)$ where $n = [G : N]$. We will use the following formula, for $u \in RG$ and S a set of representatives of the conjugacy classes of N , which is proved (for the split case but the argument holds generally) in [13, 41.10]:

$$(2) \quad \text{tr} \rho(u) = \sum_{s \in S} u_{sG} [\text{Cen}_G(s) : \text{Cen}_N(s)] s.$$

We are ready to prove Theorem 1.

3. PROOF OF THEOREM 1

By means of contradiction we assume that G is a finite group of minimal order among those satisfying the conditions of Theorem 1 such that $U_1(\mathbb{Z}G)$ contains a torsion element u which is not conjugate in $\mathbb{Q}G^*$ to an element of G . Assume that u has minimal order among the elements satisfying this property. Applying Lemma 1 with the hypothesis that u is not conjugate in $\mathbb{Q}G$ to an element of G one deduces that there is an element $v \in \langle u \rangle$ such that $v_C < 0$ for some $C \in \mathcal{Cl}(G)$. Applying again Lemma 1 to v one deduces that v is not conjugate in $\mathbb{Q}G^*$ to an element of G and by the minimality of the order of u one can take $u = v$, that is $u_C < 0$ for some $C \in \mathcal{Cl}(G)$. This will be in contradiction with:

Main Claim: $u_C \geq 0$ for every $C \in \mathcal{Cl}(G)$. Since A is abelian, $\mathbb{Z}G^* \cap (1 + \Delta(G)\Delta(A))$ is torsion free (see [1] or [13, 31.3] under the additional hypothesis that G/A is abelian which holds here and [9] without this hypothesis). Using this it is easy to show that if v and w are two torsion units of $U_1(\mathbb{Z}G)$ such that $v \equiv w \pmod{\Delta(G)\Delta(A)}$ then $o(v) = o(w)$. We will use this several times without specific mention.

If B is a subgroup of A then B is normal in G and it is easy to see that G/B also satisfies the conditions of Theorem 1. Therefore, if B is non trivial then the minimality of the order of G implies that (ZC1) holds for G/B . We will also use this several times without specific mention. Let $\pi = \pi(G) \setminus \pi(X)$ and $\lambda = \pi(G) \setminus \pi(A)$ so that $\pi(G)$ is the disjoint union of π , λ and possibly $\{p\}$.

Set $\Delta(G) = \Delta_{\mathbb{Z}}(G, G)$ and $\Delta(A) = \Delta_{\mathbb{Z}}(A, A)$. Using that G/A is abelian and hence every torsion unit of augmentation one in $\mathbb{Z}(G/A)$ is an element of G/A and the Whitcomb argument [13, 30.5] one deduces that there is $y \in G$ such that $u \equiv y \pmod{\Delta(G)\Delta(A)}$. Then $u_{\tau} \equiv y_{\tau} \pmod{\Delta(G)\Delta(A)}$ for every $\tau \subseteq \pi(G)$. In particular $y_q \in A_q$ if $q \in \pi$. Moreover if $\lambda_1 = \{p\} \cup \lambda$ and $z_0 = y_{\lambda_1}$ then $\pi(z_0) \subseteq \lambda_1$. Since $A_p X$ is a Hall λ_1 subgroup of G then z_0 is conjugate to an element of $A_p X$ [12, 9.1.3]. Conjugating u by this element we assume that $z_0 \in A_p X = (A_p X_p) \times X_{\lambda}$ and therefore $y_p \in A_p X_p$ and $y_{\lambda} \in X_{\lambda} = X_{p'}$. *Claim 1.* $\pi(A) \subseteq \pi(u)$.

This is basically a consequence of [3, 2.2]. We include an elementary (slightly different) proof for the sake of completeness. By means of contradiction let $q \in \pi(A) \setminus \pi(u)$. Let $\bar{G} = G/A_q$ and use the bar notation for reduction modulo A_q . Since $A_q \neq 1$, (ZC1) holds for $\bar{G} = G/A_q$, so that $\bar{u}_D \geq 0$ for every $D \in \mathcal{Cl}(\bar{G})$. However we know that $u_C < 0$ for some $C \in \mathcal{Cl}(G)$. Let $g \in C$. By [13, 38.11], if $h \in G$ is an element of C whose order is multiple of q then $u_{hC} = 0$. This shows that $g_q = 1$. However we are going to show that if $h_q = 1$ and $\bar{h} \sim \bar{g}$ then $h \sim g$, so that $0 \geq \bar{u}_{\bar{C}} = u_C < 0$, a contradiction. Indeed, if $\bar{h} \sim \bar{g}$ then $h \sim ag$, $a \in A_q$. Since $o(h)$ is not multiple of q , one has that $o(h) = o(g)$ and therefore $(a, g) \neq 1$. Write $a^g = a^j$ and $q^m = o(a)$. As in the proof of Lemma 3 we have that $j \not\equiv 1 \pmod{q}$. Let $k(j-1) \equiv 1 \pmod{q^m}$. Then $a^k g a^{-k} = a^{k(j-1)} g = ag \sim h$, and this finishes the proof of Claim 1. *Claim 2.* $(y_t, A_{\pi}) = 1$ for every $t \in \pi(G)$.

The statement is obvious if $t \in \pi$ because $y_t \in A_{\pi}$ and A_{π} is abelian. Otherwise $y_t \in \langle y_{\pi(X)} \rangle$ and therefore it is enough to show that $(y_{\pi(X)}, A_{\pi}) = 1$. On the other hand $A_{\pi} = \prod_{q \in \pi} A_q$ and thus to prove Claim 2 one only needs to show that $(y_{\pi(X)}, A_q) = 1$ for every $q \in \pi$. Let $q \in \pi$. Then $u_q \neq 1$, by Claim 1. On the other hand, $u_{\pi(X)} u_q \equiv y_{\pi(X)} y_q \pmod{\Delta(G)\Delta(A)}$ and hence $q | o(u_{\pi(X)}) o(u_q) = o(u_{\pi(X)} u_q) = o(y_{\pi(X)} y_q)$. Since $y_q \in A_q$, applying Lemma 3 we deduce that $(y_{\pi(X)} y_q, A_q) = 1$, or equivalently $(y_{\pi(X)}, A_q) = 1$. This proves Claim 2. *Claim 3.* $\pi(A) = \pi(u)$ and $u \equiv 1 \pmod{\Delta(G, M)}$, where $M = \text{Cen}_{AX_p}(A_{\pi})$. By Claim 2, $z = y_{\lambda} \in \text{Cen}_{X_{p'}}(A_{\pi})$. Recall that we have been careful to pick $z_0 \in A_p X = (A_p X_p) \times X_{\lambda}$ and therefore $z \in X_{\lambda}$. Thus z commutes with A_p, A_{π} and X and therefore is a central element of G . Furthermore $uz^{-1} \equiv y_{\pi(A)} \pmod{\Delta(G)\Delta(A)}$ and therefore $o(uz^{-1}) = o(y_{\pi(A)})$. Since

z is central, uz^{-1} is not conjugate in $\mathbb{Q}G^*$ to an element of G . Now the minimality of $o(u)$ implies that $o(y) = o(u) = o(uz^{-1}) = o(y_{\pi(A)}) = o(y)/o(z)$ and so $z = 1$. This shows that $y = y_{\pi}y_p \in A_{\pi}A_pX_p = AX_p$. Combining this with Claim 2 one deduces that $y \in M$ and hence $u \equiv y \equiv 1 \pmod{\Delta(G, M)}$. This establishes Claim 3.

Finally we address the proof of the Main Claim. Let $g \in C \in \mathcal{Cl}(G)$. Assume first $g \notin \text{Cen}_G(A)$. Then $B = \langle (a, g) \rangle \neq 1$ for some $a \in A$. Hence (ZC1) holds for $\bar{G} = G/B$. Using Lemma 1 there is $D \in \mathcal{Cl}(\bar{G})$ such that $\bar{u}_E = 0$ for every $D \neq E \in \mathcal{Cl}(\bar{G})$. Then $\bar{u}_{\bar{D}} = 1$, since $\bar{u}_{\bar{G}} = u_G = 1$. On the other hand $C = g^G \subseteq G'g \subseteq Ag$. Using Lemma 2 one has that $BC = C$ and this implies that $u_C = \bar{u}_{\bar{C}} \geq 0$.

Now let $g \in \text{Cen}_G(A)$ and let $N = \langle M, g \rangle$. Let $g = hx$ with $h \in \langle g \rangle_{\pi(A)}$ and $x \in \langle g \rangle_{\lambda}$. Then $h \in M$ and so $N = \langle M, x \rangle$ and $x^{h_0} \in X$ for some $h_0 \in G$. Since $x \in \text{Cen}_G(A)$ then $x^{h_0} \in \text{Cen}_X(A) \subseteq Z(G)$. Therefore $x \in Z(G)$ and we conclude that $N = \langle M, x \rangle = A_{\pi} \times \text{Cen}_{A_pX_p}(A_{\pi}) \times \langle x \rangle$, a nilpotent group. Furthermore N_q is cyclic for every $p \neq q \in \pi(N)$. Let $n = [G : N]$ and let $\rho : \mathbb{Q}G \rightarrow M_n(\mathbb{Q}N)$ as explained at the end of section 2. Notice that ρ restricts to a group homomorphism $\mathbb{Z}G^* \cap (1 + \Delta(G, N)) \rightarrow \text{SGL}_n(\mathbb{Z}N)$. By Claim 3, $u \in \mathbb{Z}G^* \cap (1 + \Delta(G, M)) \subseteq \mathbb{Z}G^* \cap (1 + \Delta(G, N))$ and hence $\rho(u)$ is a torsion unit of $\text{SGL}_n(\mathbb{Z}N)$. In the previous paragraph we have checked that N satisfies the hypothesis of Theorem 7. Thus there is $\alpha \in \text{GL}_n(\mathbb{Q}N)$ such that $\rho(u) = \alpha^{-1}d\alpha$ for d a diagonal matrix formed by elements of N . This implies that $\text{tr}\rho(u) = \text{tr}d$ is of the form $\sum_{b \in N} k_b b$ with k_b non negative integers. By using (1) and (2) one has that $u_D \geq 0$ for every $D \in \mathcal{Cl}(G)$ such that $D \subseteq N$, and in particular $u_C \geq 0$. This completes the proof of the Main Claim and of Theorem 1. $\square$

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