The Recovery Question for Incidence Rings

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Abstract

Let P and P' be finite preordered sets, and let R be a ring for which the number of nonzero summands in a direct decomposition of the regular module R_R is bounded. We show that if the incidence rings $I(P,R)$ and $I(P',R)$ are isomorphic as rings, then P and P' are isomorphic as preordered sets. We give a stronger version of this result in case P and P' are partially ordered. We show that various natural extensions of these results fail. Specifically, we show that if $\{P_j \mid j \in \Omega\}$ is any collection of (locally finite) preordered sets then there exists a ring S such that the incidence rings $\{I(P_j,S) \mid j \in \Omega\}$ are pairwise isomorphic. Additionally, we verify that there exists a finite dimensional algebra R and locally finite, nonisomorphic partially ordered sets P and P' for which $I(P,R) \simeq I(P',R)$.

Throughout this article P will denote a locally finite preordered set; so P is a set equipped with a reflexive, transitive relation such that for any two elements $x, y \in P$, the set $\{z \in P \mid x \leq z \leq y\}$ is finite. R will denote an associative unital ring. The *incidence ring of P with coefficients in R*, denoted by $I(P,R)$ (or simply by A throughout this article), is the ring of functions $\{f : P \times P \to R \mid f(x,y) = 0 \ \forall x \not\leq y\}$; multiplication is given by $(fg)(x,y) = \sum_{x \leq z \leq y} f(x,z)g(z,y)$ (and is well-defined due to the local finiteness condition on P). The ring $I(P,R)$ may also be viewed as square matrices with entries in R, whose rows and columns are indexed by P, for which the (x,y) coordinate is 0 whenever $x \not\leq y$.

Given locally finite preordered sets P and P', and rings R and R', the Recovery Question for Incidence Rings is posed as follows: If the incidence rings $I(P,R)$ and $I(P',R')$ are isomorphic, under what hypotheses may we conclude that $P \simeq P'$ as preordered sets and/or $R \simeq R'$ as rings? Our investigation in this article focuses on a specific version of the recovery

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question; namely, we find various conditions on P, P', and R for which the hypothesis $I(P,R) \simeq I(P',R)$ yields the conclusion that $P \simeq P'$ as preordered sets.

A number of authors have investigated the Recovery Question for Incidence Rings. We give here in tabular form a thumbnail historical sketch of various conditions that previous authors have used to show that $I(P,R) \simeq I(P',R) \Rightarrow P \simeq P'$.

Author	**Hypotheses on Sets**	**Hypoth. on Coeff. Ring**
Stanley 1970 [6]	P, P' partially ordered	R a field
Belding 1973 [1]	P, P' preordered, with one finite	R a field
Nachev 1977 [5]	P, P' preordered	R simple right artinian
Voss 1980 [7]	P, P' partially ordered	R indecomposable mod $J(R)$
ibid.	P, P' preordered	R simple artinian mod $J(R)$
ibid.	P, P' preordered	R indec. semiperfect (see below)
Haack 1984 [3]	P, P' finite partially ordered	R semiperfect
Dăsc./VanWyk 1996 [2]	P, P' finite preordered	R semiprime noetherian
ibid.	P, P' finite preordered	R commutative noetherian

We note that in a parenthetical remark made in [7], Voss actually extends his solution to the Recovery Question to coefficient rings which are finite direct sums of indecomposable semiperfect rings, each of whose associated preordered sets is pairwise isomorphic. Each of the above mentioned articles contains many results in addition to those cited. Moreover, various examples are given in these articles to show that the given hypotheses cannot be eliminated.

In this article we make two types of contributions to the investigation of the Recovery Question. In Section 1 we extend all of the above results, in the situation where P and P' are finite. We do so by establishing the following solutions to two versions of the Recovery Question.

Solution 1. (cf. Theorem 1.12) Let R be a ring with the property that the integer $max(\{n \mid \text{there exist nonzero right ideals } K_1, K_2, \ldots, K_n \text{ with } R = K_1 \oplus K_2 \oplus \cdots \oplus K_n\})$ exists, and let P and P' be finite preordered sets. If $I(P,R) \simeq I(P',R)$ as rings, then $P \simeq P'$ as preordered sets.

Numerous classes of rings have the property mentioned in Solution 1; we say such rings have *finite summand length.* (It is not hard to show that this property is left/right symmetric.) Such classes include rings with Goldie dimension (e.g. Noetherian rings), and semiperfect rings (e.g. finite dimensional algebras). In particular, Solution 1 extends [2, Theorem 2.4], even more so than is needed to answer in the affirmative the conjecture made at the end of that article. Along the way to developing the tools needed to prove Theorem 1.12, we establish

Solution 2. (cf. Theorem 1.6) Let R be a ring with the property that there are only finitely many direct summands of the right regular module R_R which are in addition two-sided ideals of R, and let P and P' be finite partially ordered sets. If $I(P,R) \simeq I(P',R)$ as rings, then $P \simeq P'$ as partially ordered sets.

Our method of attack used in establishing these two solutions is somewhat similar in flavor to that used by Voss [7]. As we do not necessarily work over semiperfect coefficient rings, we are not afforded the luxury of working with a canonical set of indecomposable projective modules (which proved so helpful to Voss and others). Instead, we focus on a particular set of projective right ideals $D(T)$ for a general ring T, and then show that there is a strong relationship between $D(R)$ and $D(I(P,R))$. This relationship allows us to produce an isomorphism between the direct products of certain preordered (resp. partially ordered) sets, which then puts us in position to use a 'cancellation' theorem of Lovasz [4, Theorem 4.3] to establish the desired result.

Our second contribution surfaces in Section 2, where we show that natural extensions of various solutions to the Recovery Question (our own, as well the solutions of others mentioned above) fail. We show first (Proposition 2.1) that Solution 2 cannot be extended to preordered sets. Next we show (Proposition 2.3) that if $\{P_j \mid j \in \Omega\}$ is any collection of locally finite preoredered sets then there exists a ring S such that the incidence rings $\{I(P_j, S) \mid j \in \Omega\}$ are pairwise isomorphic, thereby demonstrating that any solution to the Recovery Question must contain some stipulation on the structure of the coefficient rings. Finally, we answer a question posed at the end of [7] by demonstrating (Proposition 2.4) the existence of a finite dimensional algebra R and locally finite, nonisomorphic partially ordered sets P and P' for which $I(P,R) \simeq I(P',R)$.

1 Solutions to the Recovery Question

We begin with some notation and observations. For a ring T we let $\mathrm{Id}(T)$ denote the set of two-sided ideals of T, while $\mathrm{Id}(T_T)$ denotes the set of right ideals of T. We let

$$D(T) = \{K \in \mathrm{Id}(T) | K \text{ is a direct summand of } T_T\} = \{eT | e = e^2 \in T \text{ and } Te \subseteq eT\}.$$

In the sequel we will often view $D(T)$ as a partially ordered set with relation $\subseteq$. It is not hard to show that $D(T)$ is closed under finite sums. For two preordered sets P and P' with $x \in P$ we let

$$\hat{x} = \{y \in P | x \leq y \leq x\}$$

$$\hat{P} = \{\hat{y} | y \in P\} \quad \text{(the partially ordered set associated to } P)$$

$$\mathrm{AntiHom}(P,P') = \text{ the set of preordered anti-homomorphisms from } P \text{ to } P'.$$

The set $\mathrm{AntiHom}(P,P')$ is preordered, where for $F, G \in \mathrm{AntiHom}(P,P')$ we set $F \leq G$ in case $F(x) \leq G(x)$ for all $x \in P$. The preordered sets P and P' are isomorphic in case there exists a set bijection $H : P \to P'$ such that both H and H^{-1} preserve the respective orders on P and P'. If $|\hat{x}| = n$, then for any ring T we have $I(\hat{x}, T) \simeq M_n(T)$, the ring of $n \times n$ matrices with coefficients in T. For a preordered set P and a ring R we set $A = I(P,R)$. If X is a subset of P, e_X denotes the idempotent of A given by $e_X(x,y) = 1$ if $x = y \in X$, and $e_X(x,y) = 0$ otherwise. For $x \in P$ we denote $e_{\{x\}}$ by e_x.

Our first task is to identify the set of two-sided ideals of $A = I(P,R)$ in case P is finite. Given a map $\Psi : P \times P \to \mathrm{Id}(R_R)$, we let

$$I(\Psi) = \{\alpha \in A : \alpha(x,y) \in \Psi(x,y) \text{ for all } x \leq y \text{ in } P\}.$$

Lemma 1.1 *Let P be a preordered set, and let $\Psi : P \times P \to \mathrm{Id}(R_R)$ be a map.*

1. *$I(\Psi) \in \mathrm{Id}(A_A)$ if and only if $\Psi(x,y) \subseteq \Psi(x,y')$ for every $x \leq y \leq y'$ in P.*
2. *$I(\Psi) \in \mathrm{Id}(A)$ if and only if $\Psi(x,y) \in \mathrm{Id}(R)$ and $\Psi(x,y) \subseteq \Psi(x',y')$ for every $x' \leq x \leq y \leq y'$ in P.*
3. *If P is finite, then every two-sided ideal of A is of the form $I(\Psi)$ for some map $\Psi : P \times P \to \mathrm{Id}(R_R)$ satisfying the conditions of (2)*

Proof. Statements (1) and (2) are straightforward to verify, using the definition of multiplication in the incidence ring $A = I(P,R)$.

For statement (3), assume that I is a two-sided ideal of A. For every $x \leq y$ in P let $\Psi(x,y) = \{\alpha(x,y) | \alpha \in I\}$. Clearly $\Psi(x,y)$ is a two-sided ideal of R, and $I \subseteq I(\Psi)$. Let $\alpha \in I(\Psi)$. Then for every pair $x \leq y \in P$ there is $\alpha_{x,y} \in I$ such that $\alpha_{x,y}(x,y) = \alpha(x,y)$. By the finiteness of P we may then write $\alpha = \sum_{x \leq y} e_x \alpha_{x,y} e_y$, which is an element of I. Therefore $I = I(\Psi)$ and hence Ψ satisfies the conditions of (2). $\diamond$

Having associated the two-sided ideals of A with certain maps from $P \times P$ to $\mathrm{Id}(R)$, we now determine additional properties of such maps which ensure that the associated ideal of A is in fact an element of $D(A)$. Specifically, given any map $F : \hat{P} \to \mathrm{Id}(R_R)$ we define

$$\Psi_F : P \times P \to \mathrm{Id}(R_R) \text{ by } \Psi_F(x,y) = F(\hat{x}), \text{ and}$$
$$J(F) = I(\Psi_F).$$

From the matrix point of view, $J(F)$ is the collection of matrices in $I(P,R)$ for which the entry in the (x,y) coordinate is taken from the right ideal $F(\hat{x})$. In particular, the entries in a given row of an element of $J(F)$ belong to the fixed right ideal $F(\hat{x})$.

Lemma 1.2 *Let P be a preordered set. Let F be a map from $\hat{P}$ to $\mathrm{Id}(R_R)$. Then*

1. *$J(F)$ is a right ideal of A.*
2. *$J(F)$ is a two-sided ideal of A if and only if $\mathrm{Im}\, F \subseteq \mathrm{Id}(R)$ and F is an anti-homomorphism of partially ordered sets.*
3. *If P is finite, then $D(A) \simeq \{J(F) | F \in \mathrm{AntiHom}(\hat{P}, D(R))\}$.*

Proof. (1) and (2) are consequences of Lemma 1.1.

(3) Let $F \in \mathrm{AntiHom}(\hat{P}, D(R))$. For every $x \in \hat{P}$, let $F_1(x) \in \mathrm{Id}(R_R)$ be a complement of $F(x)$, so that $F(x) \oplus F_1(x) = R$. Then it is easy to check that $J(F) \oplus J(F_1) = A$, so that $J(F) \in D(A)$.

Conversely, let $J = fA \in D(A)$; so $f = f^2 \in A$, and $Af \subseteq fA$. For every $X \in \hat{P}$ let $f_X \in M_X(R)$ be given by $f_X(x,y) = f(x,y)$ for every $x,y \in X$. If $x,y \in X$, $f_X^2(x,y) = \sum_{z \in X} f_X(x,z) f_X(z,y) = \sum_{x \leq z \leq y} f(x,z) f(z,y) = f(x,y) = f_X(x,y)$. Thus f_X is an idempotent of $M_X(R)$. Moreover $f_X M_X(R) \in D(M_X(R))$ because if $r \in M_X(R)$, then

there is $\alpha \in A$ such that $re_X f = f\alpha$ and hence $rf_X = f_X\alpha_X$ (where α_X is defined similarly). Since the ideals of $M_X(R)$ each arise from an ideal of R, we have that there is two-sided ideal K_X of R such that $f_X M_X(R) = M_X(K_X)$. But $M_X(K_X)$ is then also a direct summand of $M_X(R)_{M_X(R)}$, and hence $e_x M_X(K_X)$ is a direct summand of $e_x M_X(R)_{M_X(R)}$ for every $x \in X$. Since the functor given by $e_x M_X(-)$ is precisely the equivalence functor between $Mod - M_X(R)$ and $Mod - R$, we conclude that K_X is a direct summand of R_R. Thus $K_X = f_X^1 R$ for some idempotent f_X^1 of R such that $Rf_X^1 \subseteq f_X^1 R$. Consequently we have $f_X M_X(R) = M_X(K_X) = M_X(f_X^1 R) = g_X \cdot M_X(R)$, where g_X is the scalar matrix in $M_X(R)$ with f_X^1 on the diagonal. In particular, f_X generates the same right (in fact, two-sided) ideal of $M_X(R)$ as the scalar g_X .

Now let $F : \hat{P} \to D(R)$ be given by $F(X) = f_X^1 R$. We define $f_1 \in A$ by setting

$$f_1(x, y) = f_X^1 \text{ if } \hat{x} = \hat{y} = X, \text{ while } f_1(x, y) = 0 \text{ otherwise.}$$

So f_1 is simply the diagonal blocks of f. As such, using the finiteness of P we may write $f_1 = \sum_{\{(x,y)|\hat{x}=\hat{y}\}} e_x f e_y$, so that $f_1 \in fA$ as fA is two-sided (being an element of $D(A)$). In particular, since f is idempotent we get that $ff_1 = f_1$. Again using the finiteness of P we have that $f - f_1$ is nilpotent (necessarily of degree $\leq |P|$). Expanding the equation $(f-f_1)^t = 0$ and using that $ff_1 = f_1$ yields that $f = f^t \in f_1 A$. Thus we have that $fA = f_1 A$. Now let f_2 be the block-diagonal element of A whose entry in the $X - X$ component is g_X. By the discussion in the previous paragraph we have that $f_1 A = f_2 A$, so that $fA = f_2 A$. But the diagonal form of f_2 immediately yields that $f_2 A = J(F)$, so that finally we have shown that $J = fA = f_2 A = J(F)$ is of the required form. That $F \in \text{AntiHom}(\hat{P}, D(R))$ follows from part (2). $\diamond$

Definition 1.3 *Let P be a partially ordered set. We define the partially ordered subset P_1 of P by setting*

$$P_1 = \{x \in P | \text{ there exists } x' < x \text{ with the property that } y \leq x' \text{ for every } y < x\}.$$

In other words, x is in P_1 in case there exists a unique maximal element x' in X below x.

If T is any ring then as noted above $D(T)$ is a partially ordered set under inclusion. In the sequel we will focus much attention on the partially ordered subset $D(T)_1$ of $D(T)$. We first prove a general result about such subsets.

Lemma 1.4 *Let P and Q be two partially ordered sets, such that P is finite, and such that Q has a unique minimal element 0. Let $H : P \times Q \to \text{AntiHom}(P, Q)$ be the map given by setting $H(p, q)(x) = q$ if $x \leq p$ and $H(p, q)(x) = 0$ otherwise. Then H restricts to an isomorphism of partially ordered sets*

$$H : P \times Q_1 \simeq \text{AntiHom}(P, Q)_1.$$

In particular, for every $(p, q) \in P \times Q_1$, the unique maximal element $H(p, q)' < H(p, q)$ is given by setting $H(p, q)'(x) = q$ if $x < p$, $H(p, q)'(p) = q'$, and $H(p, q)'(x) = 0$ otherwise.

Proof. First we check that $H = H(p,q) \in \mathrm{AntiHom}(P,Q)$. Indeed, suppose $x \leq y$ in P. Then either $y \leq p$, in which case $H(y) = q = H(x)$, or $y \not\leq p$, in which case $H(y) = 0 \leq H(x)$.

It is clear that the restriction of H to $P \times (Q \setminus \{0\})$ is injective and preserves the order. So we must check that $H(P \times Q_1) = \mathrm{AntiHom}(P,Q)_1$, and that the inverse map preserves the order. To this end, suppose that $(p,q) \in P \times Q_1$. Let $H' : P \to Q$ be given by $H'(x) = q$ if $x < p$, $H'(p) = q'$, and $H'(x) = 0$ if $x \not\leq p$. Then $H' < H = H(p,q)$. We verify that H' satisfies the desired properties. So let $G < H$ in $\mathrm{AntiHom}(P,Q)$; we show $G \leq H'$. Now $G(x) \leq H(x) = 0$ if $x \not\leq p$ and hence $G(x) = 0$ in this case. Moreover, $G(p) \leq H(p) = q$. Assume that $G(p) = q$. If $x \leq p$, then $q = G(p) \leq G(x) \leq H(p) = q$; but this gives $G = H$, a contradiction. Therefore $G(p) < q$, and hence $G(p) \leq q'$. Finally, if $x < p$, then $G(x) \leq H(x) = q$. Thus $G \leq H'$ and this proves that $H \in \mathrm{AntiHom}(P,Q)_1$.

Conversely, let $H \in \mathrm{AntiHom}(P,Q)_1$. Let p and p_1 be two different maximal elements of $\{x \in P : H(x) \neq 0\}$. Let $F_1, F_2 : P \to Q$, be given by

$$F_1(x) = F_2(x) = H(x) \text{ for } p \neq x \neq p_1 \text{ and}$$

$$F_1(p) = H(p), \quad F_1(p_1) = 0, \qquad F_2(p) = 0, \quad F_2(p_1) = H(p_1).$$

Since $F_1 < H$ and $F_2 < H$ we have $F_1 \leq H'$ and $F_2 \leq H'$. Then $H(x) = F_1(x) \leq H'(x) \geq F_2(x) = H(x)$ for $p \neq x \neq p_1$, $H(p) = F_1(p) \leq H'(p) \leq H(p)$, and $H(p') = F_2(p') \leq H'(p') \leq H(p')$. But this gives $H = H'$, a contradiction. Therefore there is an element $p \in P$, such that $H(p) \neq 0$ and $H(x) = 0$, for every $x \not\leq p$.

Let $F : P \to Q$ be given by $F(x) = H(x)$ if $x \neq p$ and $F(p) = 0$. Since $F \in \mathrm{AntiHom}(P,Q)$ and $F < H$, then $F \leq H'$. Therefore $H(x) \geq H'(x) \geq F(x) = H(x)$, and hence $H(x) = H'(x)$, for every $x \neq p$. Since $H' < H$, then $H'(p) < H(p)$.

Set $q = H(p)$. If $H \neq H(p,q)$, there is $x < p$ such that $H(x) > q$. Assume that x is maximal with that condition. Let $F \in \mathrm{AntiHom}(P,Q)$ be given by $F(y) = H(y)$, if $y \neq x$ and $F(x) = q$. Then $F < H$ and hence $F \leq H'$. Therefore $q = F(p) \leq H'(p) \leq H(p) = q$. Thus $H' = H$ a contradiction. We conclude that $H = H(p,q)$.

We now show that $q \in Q_1$. We prove that $H'(p)$ has the desired properties; i.e., we prove that $H'(p) = q'$. If $x < q$, let $F \in \mathrm{AntiHom}(P,Q)$ be given by $F(y) = H(y)$ if $y \neq p$ and $F(p) = x$. Since $F < H$, then $F \leq H'$ and hence $x \leq H'(p)$.

Now that we have shown that H is a set bijection which preserves the order, we need only show that H^{-1} preserves the order as well. So suppose that $H(p_1,q_1) \leq H(p_2,q_2)$. Then on applying each of these antihomomorphisms to $p_1 \in P$ we get $q_1 = H(p_1,q_1)(p_1) \leq H(p_2,q_2)(p_1)$. But $q_1 \neq 0$, so $H(p_2,q_2)(p_1) \neq 0$, which by definition of $H(p_2,q_2)$ gives that $p_1 \leq p_2$, and that $q_1 \leq q_2$. $\diamond$

Lemmas 1.2 and 1.4 now yield

Proposition 1.5 *Let P be a finite preordered set, and let R be any ring. Then there is an isomorphism of partially ordered sets*

$$H : \hat{P} \times D(R)_1 \simeq D(A)_1.$$

Specifically, for $(X,K) \in \hat{P} \times D(R)_1$ we set $H(X,K) = J(F_{X,K})$, where $F_{X,K} \in AntiHom(\hat{P}, D(R)_1)$ is defined by setting $F_{X,K}(W) = K$ if $W \leq X$, $F_{X,K}(W) = 0$ otherwise .

In turn, Proposition 1.5 allows us to establish one of the two Solutions to the Recovery Question mentioned in the introduction.

Theorem 1.6 *Let P and P' be finite partially ordered sets, and let R be a ring for which $D(R)$ is finite. If the incidence rings $I(P,R)$ and $I(P',R)$ are isomorphic, then P and P' are isomorphic as partially ordered sets.*

Proof. As P and P' are partially ordered we have $P = \hat{P}$ and $P' = \widehat{P'}$. Let A denote $I(P,R)$ and let A' denote $I(P',R)$. It is straightforward to show that a ring isomorphism between A and A' yields an isomorphism of the corresponding partially ordered sets $D(A)_1$ and $D(A')_1$. By Lemma 1.5 this yields an isomorphism of partially ordered sets $P \times D(R)_1 \simeq P' \times D(R)_1$. Now an application of [4, Theorem 4.3] yields the desired isomorphism between P and P'. $\diamond$

As is apparent from the proof of Lemma 1.2, $D(T) \simeq D(M_n(T))$ for any ring T and any matrix ring $M_n(T)$. Thus, as $M_n(T) = I(\hat{x}, T)$ for $|\hat{x}| = n$, $D(-)$ cannot be used to recover preordered sets. Therefore, in order to establish a result similar to that given in Theorem 1.6 in the more general context of preordered sets, we must utilize the more subtle structure $D^*(-)$ (defined below). The price we pay for weakening the hypothesis on the ordered sets is the necessity of imposing additional structure on the underlying coefficient ring.

Definition 1.7 *For any ring T and any right T-module M_T we define the* **summand length of** *M, denoted $w(M_T)$, by setting*

$$w(M_T) = \max\{n | M = \oplus_{i=1}^{n} M_i, \text{ for some nonzero submodules } M_i \leq M\}.$$

We set $w(M_T) = \infty$ if no such maximum exists. We say the ring T has **finite summand length** *in case $w(T_T)$ is finite.*

We note that the definition of *finite summand length* for a ring T given above is left/right symmetric. To see this, we observe that any direct decomposition of the right regular module T_T having n nonzero summands gives rise to a complete set of n orthogonal idempotents in T. This set of idempotents can then be used to give a direct decomposition of the left regular module ${}_TT$ having n nonzero direct summands. The desired conclusion follows by symmetry.

Lemma 1.8 *For any ring T, if T has finite summand length, then $D(T)$ is finite.*

Proof. We denote $w(T_T)$ by N. Since any chain of proper inclusions of length n of nonzero elements of $D(T)$ gives rise to a decomposition of T_T containing $n+1$ summands, we conclude that the partially ordered set $D(T)$ has finite height (in fact, height $\leq N$). Thus to show that $D(T)$ is finite it is enough to show that for every $K \in D(T)$, the set of minimal elements of $\{J \in D(T) | K \subset J\}$ is finite. Indeed, if we denote this set of minimal elements by $\{J_i | i \in \Omega\}$, then we show that this set contains at most N elements, by showing that this set can be used to generate an equal-sized set of nonzero pairwise orthogonal idempotents in T. Let

$K = gT$, $J_i = e_iT$ for idempotents g, $\{e_i\}_{i\in\Omega}$. Since $K \subset J_i$ we have $e_ig = g$ for all i. With this it is easy to show that each $e_i' = e_i - ge_i$ is idempotent. Further, it is easy to show that $e_iT \cap e_jT = e_ie_jT \in D(T)$; the minimality property of these ideals then implies that $e_ie_jT = gT$ for $i \neq j$. It follows that e_i' and e_j' are in fact orthogonal for $i \neq j$. Since $e_i \neq g$ we have that e_i' is nonzero for all i, and so $\{e_i'\}_{i\in\Omega}$ is a set of nonzero pairwise orthogonal idempotents. But a set of more than N nonzero pairwise orthogonal idempotents in T would yield a decomposition of T_T in violation of the hypothesis that $w(T_T) = N$, so

that $\mid \Omega \mid \leq N$ as desired. $\diamond$

Definition 1.9 *Let T be any ring with finite summand length. We define*

$$D^*(T) = \{(K, i) | K \in D(T)_1 \text{ and } 1 \leq i \leq w((K/K')_T)\}.$$

(Here K' is the unique maximal element of $D(T)$ below K, as given in Definition 1.3.) We note that $w((K/K')_T)$ is indeed an integer, as K/K' is isomorphic to a direct summand of T_T. We define a preorder on $D^*(T)$ by setting

$$(K, i) \leq (L, j) \Leftrightarrow K \subseteq L.$$

For any positive integer s, let $\mathbf{N}_s$ denote the set $\{1, 2, \ldots, s\}$.

Lemma 1.10 *Let P be a finite preordered set, and let R be a ring with finite summand length. If $K \in D(R)_1$ and $X \in \hat{P}$, then*

$$|X| \cdot w(K/K'_R) = w(H(X, K)/H(X, K)'_A).$$

In particular, there exists a bijection

$$\phi_{X,K} : X \times \mathbf{N}_{w(K/K')} \to \mathbf{N}_{w(H(X,K)/H(X,K)')}.$$

Proof. Let $h(X, K)$ denote the right ideal $e_XH(X, K)$ of A. By Lemma 1.4 one can show that $H(X, K) = (\sum_{Y<X} H(Y, K)) \oplus h(X, K)$, and that the unique maximal element $H(X, K)'$ below $H(X, K)$ is given by $H(X, K)' = (\sum_{Y<X} H(Y, K)) \oplus h(X, K')$. Therefore, $H(X, K)/H(X, K)' \simeq h(X, K)/h(X, K')$.

Now let $J = I(P, K')$. Since $h(X, K)J \subset h(X, K')$, then $h(X, K)/h(X, K')$ is a right A/J-module. Moreover, if $B = I(P, R/K')$, then $(h(X, K)/h(X, K'))_{A/J} \simeq h(X, K/K')_B$ and it follows that

$$w((h(X, K)/h(X, K'))_A) = w((h(X, K)/h(X, K'))_{A/J}) = w(h(X, K/K')_B).$$

Consequently, $w(H(X, K)/H(X, K)'_A) = w(h(X, K/K')_B)$. Since K/K' is isomorphic to a direct summand of K it is in fact isomorphic to a right ideal of R; thus it suffices to prove that if L is a right ideal of R, then $w(h(X, L)_A) = |X|w(L_R)$.

If $L = \oplus_{i=1}^n L_i$, then $h(X, L) = \oplus_{x\in X} \oplus_{i=1}^n e_xh(X, L_i)$ since e_X acts like an identity on the left of elements of $h(X, L)$. This shows that $w(h(X, L)) \geq |X|w(L)$. Now suppose $w(h(X, L)) = n$ and so we may write $h(X, L) = \oplus_{i=1}^n L_i$ where each L_i is a nonzero right

ideal of A. Again, since e_X is an identity on the left of each L_i, we have $L_i = \oplus_{x\in X} e_x L_i$ and so $h(X,L) = \oplus_{i=1}^n \oplus_{x\in X} e_x L_i$. Since $n = w(h(X,L))$, it follows that, for every $i = 1,\ldots,n$, there is exactly one $x_i \in X$ such that $L_i = e_{x_i} L_i$. (It may be the case that two such L_i share the same x.) For every $i = 1,\ldots,n$, let $J_i = \{\alpha(x_i,x_i) : \alpha \in L_i\}$. It is clear that each J_i is a right ideal of R and that, for every $x \in X$, $L = \sum_{i,x_i=x} J_i$. We claim that this sum is direct. To see this, suppose $\sum_{i,x_i=x} r_i = 0$, with $r_i \in J_i$, and so there are $\alpha_i \in L_i$ such that $\alpha_i(x,x) = r_i$. Since $\alpha_i \in L_i = e_{x_i} L_i$, it follows that $\sum_{i=1}^n \alpha_i e_x = 0$. Since the original sum $\oplus_{i=1}^n L_i$ is direct, we have each $\alpha_i e_x = 0$ and so each $r_i = 0$ and the claim holds. In particular, the number m_x of the i, $i = 1,\ldots,n$ such that $x_i = x$ satisfies $m_x \le w(L)$. Consequently, $h(X,L) = \oplus_{x\in X} e_x h(X,L) = \oplus_{x\in X} \oplus_{i,x_i=x} e_x h(X,J_i)$. Once we show that $e_{x_i} h(X,J_i) = L_i$ for every i, the proof will be complete because then we shall have $w(h(X,L)) = \sum_{x\in X} m_x \le |X| w(L)$, as desired. To show the equality, let $\alpha \in e_{x_i} h(X,J_i)$. Then, for $y \ge x_i$, there are $\beta_y \in L_i$ such that $\alpha(x_i,y) = \beta_y(x_i,x_i)$ and so $\alpha = \sum_{y\ge x} \beta_y e_{x_i,y} \in L_i$ which shows that $e_{x_i} h(X,J_i) \subset L_i$. However, since $e_{x_i} h(X,J_i)$ and L_i are summands of $h(X,L)$, we see from the Modular law that $L_i = e_{x_i} h(X,J_i)$. $\diamond$

Proposition 1.11 *If P is any finite preordered set, and R is a ring with finite summand length, then $P \times D^*(R) \simeq D^*(A)$.*

Proof. Let H denote the isomorphism given in Proposition 1.5. For every $X \in \hat{P}$ and $K \in D(R)_1$, let $\phi_{X,K} : X \times \mathbf{N}_{w(K_R)} \to \mathbf{N}_{w(h(X,K)_A)}$ be
the map described in Lemma 1.10. We now define $\Phi : P \times D^*(R) \to D^*(A)$ by setting

$$\Phi(x,(K,i)) = (H(\hat{x},K), \phi_{\hat{x},K}(x,i)).$$

Using the fact that H is an isomorphism of partially ordered sets and that $\phi_{X,K}$ is a bijection, it is straightforward to check that Φ is indeed an isomorphism of preordered sets. $\diamond$

We now have all the tools required to establish our second version of the Recovery Question.

Theorem 1.12 *Let P and P' be two finite preordered sets, and let R be a ring with finite summand length. If $I(P,R) \simeq I(P',R)$ as rings, then $P \simeq P'$ as preordered sets.*

Proof. Let A denote $I(P,R)$ and let A' denote $I(P',R)$. It is straightforward to show that a ring isomorphism between A and A' yields an isomorphism of the corresponding preordered sets $D^*(A)$ and $D^*(A')$. By Proposition 1.11, this gives an isomorphism of preordered sets $P \times D^*(R) \simeq P' \times D^*(R)$. Now an application of [4, Theorem 4.3] yields the desired isomorphism between P and P'. $\diamond$

We note that Theorem 1.12 answers the question posed at the end of [2]; specifically, we have shown that the Recovery Question has a positive solution when the preordered sets are finite, and the coefficient ring is right or left noetherian.

2 Non-extendability of Solutions

In the second section of this article we present three constructions which demonstrate the failure of possible extensions of various solutions of the Recovery Question.

Our first non-extendability result shows that the hypotheses of Theorem 1.12 cannot be replaced by the weaker hypotheses of Theorem 1.6.

Proposition 2.1 *There exists a ring S having finite $D(S)$ and there exist nonisomorphic preordered sets P and P' such that $I(P,S) \simeq I(P',S)$.*

Proof. Let k be any unital ring, and let S denote the ring $RFM(k)$ of countably infinite row-finite matrices over k. It is well-known that S has the property that $S \simeq M_2(S)$, which yields that $I(P,S) \simeq I(P',S)$ where $|P| = 1$ and $P' = \hat{x}$ where $|\hat{x}| = 2$. Furthermore, since S has only one nontrivial two-sided ideal, $D(S)$ is finite. $\diamond$

Prior to presenting our second non-extendability result (Proposition 2.3 below), we review some of the ideas introduced in [7]. Let X be any locally finite preordered set, and let $a \in X$. We define

$$\hat{X}^a = \{(x_n) \in \prod_{n \in \mathbf{N}} X \mid \exists N \text{ with } x_n = a \ \forall n \geq N\}.$$

In [7, Proposition 1] Voss shows that $\hat{X}^a$ is a locally finite preordered set having the property that $X \times \hat{X}^a \simeq \hat{X}^a$ as preordered sets. Indeed, the isomorphism ϕ is given by setting $\phi(x,(x_n)) = (y_n)$, where $y_1 = x$ and $y_n = x_{n-1}$ for all $n \geq 2$.

We now let $\{X_i\}_{i \in \Omega}$ be any collection of preordered sets. For each $i \in \Omega$ let a_i denote a fixed element of X_i, and let $\bar{a}_i$ denote the element $(a_i, a_i, ...)$ of $\hat{X}_i^{a_i}$. Finally, let B denote the set

$$B = \{(b_i) \in \prod_{i \in \Omega} \hat{X}_i^{a_i} \mid b_i = \bar{a}_i \text{ for all but finitely many } i\}.$$

Lemma 2.2 *The preordered set B defined above is locally finite provided that each X_i is locally finite. Moreover, for each $i \in \Omega$ we have $X_i \times B \cong B$ as preordered sets.*

Proof. The set B inherits the preorder from the product $\prod_{i \in \Omega} \hat{X}_i^{a_i}$. Now choose $(b_i) \leq (c_i) \leq (d_i)$ in B. Then for all but finitely many $i \in \Omega$ we have that $b_i = c_i = d_i = \bar{a}_i$. The first statement now follows as each $\hat{X}_i^{a_i}$ is locally finite.

Define $\phi_i : X_i \times B \to B$ by setting $\phi_i((x_i,(b_j))) = (c_j)$, where $c_j = b_j$ in $\hat{X}_j^{a_j}$ for $j \neq i$, and $c_i = \phi((x_i, b_i))$ (using the map ϕ described above). Then it is easy to check that ϕ_i is an isomorphism of the desired type. $\diamond$

Our second non-extendability result shows that the Recovery Question must always fail without some stipulation on the structure of the underlying coefficient rings.

Proposition 2.3 *Let $\{X_i\}_{i \in \Omega}$ be any collection of locally finite preordered sets. Then there exists a ring S with the property that the incidence rings $I(X_i,S)$ and $I(X_j,S)$ are isomorphic for all $i,j \in \Omega$. Indeed, each of these incidence rings is isomorphic to S itself.*

Proof. Let B be the locally finite preordered set defined above, let T be any ring, and let S denote the incidence ring $I(B,T)$. Then for each $i \in \Omega$,

$$I(X_i, S) \cong I(X_i, I(B,T)) \cong I(X_i \times B, T) \cong I(B,T) \cong S.$$

$\diamond$

We mention that the above result generalizes an example from [2]. In [2, Example 1.1] Dăscălescu and Van Wyk construct a ring R for which the following five 'structural matrix rings over R' are isomorphic:

$$R \qquad R \oplus R \qquad M_2(R) \qquad \begin{pmatrix} R & R \\ 0 & R \end{pmatrix} \qquad \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}.$$

These structural matrix rings are precisely the incidence rings which arise from the five preordered sets (respectively)

$$\{a\} \qquad \{a,b\} \qquad \{a,b \mid a \leq b, b \leq a\} \qquad \{a,b \mid a \leq b\} \qquad \{a,b \mid b \leq a\}.$$

The final non-extendability result (Proposition 2.5) addresses the necessity of the finiteness condition on the preordered sets. The key ingredient in this result is an example of non-cancellability for certain preordered sets. In [7, Theorem 4] and a subsequent remark, Voss proves that if X, Y and Z are preordered sets for which X and Y are locally finite, and Z is finite and has the property that the connected components of Z are pairwise isomorphic, then $X \times Z \simeq Y \times Z$ implies $X \simeq Y$. We now show that the condition on the connected components of Z cannot be dropped from the hypotheses of Voss' result. We recall some notation: for x, y in a preordered set X we say that y *covers* x in case $x < y$, and there does not exist z in X with $x < z < y$.

Proposition 2.4 *There exists a finite (necessarily non-connected) partially ordered set Z and locally finite (necessarily infinite) partially ordered sets X and Y such that $X \times Z \simeq Y \times Z$ but $X \not\simeq Y$.*

Proof. Let U be any preordered set in which there exists a unique minimal element $a \in U$. Let $\hat{U}^a$ be the preordered set constructed as above, and let $\bar{a}$ denote the constant tuple $\bar{a} = (a, a, \ldots)$. Set

$$\hat{U}^a_0 = \{(a_n) \in \hat{U}^a | max\{n | a_n \neq a\} \text{ is even } \} \cup \{\bar{a}\} \quad \text{ and}$$

$$\hat{U}^a_1 = \{(a_n) \in \hat{U}^a | max\{n | a_n \neq a\} \text{ is odd } \} \cup \{\bar{a}\}.$$

We first show that $U \times \hat{U}^a_0 \simeq \hat{U}^a_1$ and $U \times \hat{U}^a_1 \simeq \hat{U}^a_0$ as preordered sets. To verify this, it suffices to prove $U \times \hat{U}^a_0 \simeq \hat{U}^a_1$, since the other isomorphism is shown symmetrically. Define $\phi : U \times \hat{U}^a_0 \to \hat{U}^a_1$ by setting $\phi(x,f)(1) = x$ and $\phi(x,f)(n) = f(n-1)$ for $n \geq 2$. That ϕ is a set bijection which preserves the underlying preordering is clear.

We next show, perhaps counter to one's intuition, that the preordered sets $\hat{U}_0^a$ and $\hat{U}_1^a$ are *not* isomorphic, as long as there exists a unique cover α of a in U.

Clearly $\bar{a}$ is the unique minimal element of both $\hat{U}_0^a$ and $\hat{U}_1^a$. Let C_0 denote the set of covers of $\bar{a}$ in $\hat{U}_0^a$; so the elements of C_0 are of the form $\widehat{\alpha_{2n}}$, the tuple whose $2n$-th coordinate is α, and all other coordinates equal a. Similarly, the set C_1 of covers of $\bar{a}$ in $\hat{U}_1^a$ are of the form $\widehat{\alpha_{2n+1}}$.

It is easy to see that every element of C_0 is covered by an element which does not cover any other element of C_0; for instance, $\widehat{\alpha_{2n}}$ is covered in such a way by $\widehat{d_{2n}}$, the tuple which is α in coordinates 1 and $2n$, and is a elsewhere. However, the element $\widehat{\alpha_1}$ of C_1 does not have this property; that is, every cover of $\widehat{\alpha_1}$ necessarily covers another element of C_1.

Since $\bar{a}$ serves as the unique minimal element for both $\hat{U}_0^a$ and $\hat{U}_1^a$, and since covers are preserved by preordered set isomorphism, the above discussion yields that $\hat{U}_0^a \not\simeq \hat{U}_1^a$ as claimed.

Now we construct the example in which cancellation fails. Let U and V be two preordered sets, with elements $a \in U$ and $b \in V$. Set $Z = U \uplus V$ (the disjoint union), $X = \hat{U}_1^a \times \hat{V}_0^b$, and $Y = \hat{U}_0^a \times \hat{V}_1^b$. Then

$$X \times Z \simeq (U \uplus V) \times (\hat{U}_1^a \times \hat{V}_0^b) \simeq (U \times \hat{U}_1^a \times \hat{V}_0^b) \uplus (V \times \hat{U}_1^a \times \hat{V}_0^b)$$

$$\simeq (\hat{U}_0^a \times V \times \hat{V}_1^b) \uplus (U \times \hat{U}_0^a \times V_1^b) \simeq (U \uplus V) \times (\hat{U}_0^a \times \hat{V}_1^b) = Y \times Z.$$

Consequently, to find an example of the desired type, it suffices to find U and V such that $X = \hat{U}_0^a \times \hat{V}_1^b \not\simeq \hat{U}_1^a \times \hat{V}_0^b = Y$. But if $U = \{a, \alpha\}$ such that $a < \alpha$ and $V = \{b\}$, then $\hat{V}_0^b = \{\bar{b}\} = \hat{V}_1^b$, while $\hat{U}_0^a \not\simeq \hat{U}_1^a$ as shown above. Then $X \simeq \hat{U}_1^a \not\simeq \hat{U}_0^a \simeq Y$, and we are done. $\diamond$

Our final non-extendability result shows that if we allow the preordered sets to be infinite, then we need not be able to recover the preordered sets, even from incidence rings with coefficients taken from a finite dimensional algebra.

Proposition 2.5 *There exist locally finite, non-isomorphic partially ordered sets P and P' and a finite dimensional algebra R such that the incidence rings $I(P, R)$ and $I(P', R)$ are isomorphic.*

Proof. Let X, Y and Z be as in Proposition 2.4 and let k be any field. Let $P = X$, let $P' = Y$, and let R denote the incidence ring $I(Z, k)$. In particular, R is a 4-dimensional algebra over k. By hypothesis we have $P \not\simeq P'$. But we do have $P \times Z \simeq P' \times Z$, and so

$$I(P, R) = I(P, I(Z, k)) \simeq I(P \times Z, k) \simeq I(P' \times Z, k) \simeq I(P', I(Z, k)) = I(P', R).$$

$\diamond$

Since the ring R given in Proposition 2.5 has finite $w(R)$ (and therefore also finite $D(R)$), we cannot extend either Theorem 1.6 or Theorem 1.12 to locally finite partially ordered sets.

At the end of [7], Voss asks whether or not his solution to the Recovery Question (where the coefficient rings are assumed to be finite direct sums of indecomposable semiperfect rings

each of whose associated preordered sets is pairwise isomorphic) extends to all artinian rings. Proposition 2.5 provides a negative answer to this question. In fact, the coefficient ring used in the Proposition is in some sense a minimal counterexample, as any algebra of dimension three or less over any field can be shown to satisfy Voss' hypotheses.

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