NON-ISOMORPHIC 2-GROUPS WITH ISOMORPHIC MODULAR GROUP ALGEBRAS

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ABSTRACT. We provide non-isomorphic finite 2-groups which have isomorphic group algebras over any field of characteristic 2, thus settling the Modular Isomorphism Problem.

1. Introduction

A classical algebraic question, included explicitly in [Bra63], states: "When do nonisomorphic groups have isomorphic group algebras?" While many positive results were proven for concrete classes, some striking counterexamples were obtained for others [Dad71, Her01]. We denote by kG the group algebra of a group G over a field k. The most classical version of this problem still open, also mentioned by Brauer [Bra63, Page 166], states:

Modular Isomorphism Problem:
Let G and H be finite p-groups and k a field of characteristic p. Does an isomorphism $kG \cong kH$ of rings imply an isomorphism $G \cong H$ of groups?

The study of modular group algebras of finite p-groups can be traced back at least to Jennings who gave a group-theoretical description of the dimension subgroups [Jen41]. The Modular Isomorphism Problem received considerable attention, but resisted a solution. Surveys on the problem and related questions include [San85, Bov98], while it was also discussed in [Bov74, Pas77, Seh78]. An almost up to date list of known results on the Modular Isomorphism Problem can be found in [HS06, EK11].

In this article we present non-isomorphic 2-groups with isomorphic group algebras over any field of characteristic 2, the presentations of which are given below. In many ways our examples are minimal as they lie close to several classes of groups for which the Modular Isomorphism Problem is known to have a positive answer: Our groups are 2-generated groups of nilpotency class 3 with a derived subgroup which is cyclic of order 4 and their order is 2^n with $n \geq 9$. Now the Modular Isomorphism Problem is known to have a positive answer for metacyclic groups [Bag88, San96], for 2-generated groups of nilpotency class 2 [BdR20] and 2-generated groups of nilpotency class 3 with elementary abelian derived subgroup [MM20] (based on [San89, Bag99]). Using an algorithm developed in [Eic08] it was also shown to hold for groups of order 2^8 [MM20]. In [EK11] the same algorithm was used to claim a positive answer also for groups of order 2^9, but as pointed out in [MM20] the programs used in the proof contained a flaw.

The Modular Isomorphism Problem remains open for various interesting classes of p-groups, including groups of nilpotency class 2 and groups of odd order.

2. The groups

We use standard group theoretical notation as for example $g^h = h^{-1}gh$ and $[g,h] = g^{-1}h^{-1}gh$ for g and h group elements. The Frattini subgroup of a group G is denoted by $\Phi(G)$ and the centralizer in G of a subset X is denoted by $C_G(X)$.

Let n and m be integers satisfying $n > m > 2$ and consider the groups given by the following presentations:

$$\begin{aligned} G &= \left\langle x, y, z \mid z = [y,x],\ x^{2^n} = y^{2^m} = z^4 = 1,\ z^x = z^{-1},\ z^y = z^{-1} \right\rangle \\ H &= \left\langle a, b, c \mid c = [b,a],\ a^{2^n} = b^{2^m} = c^4 = 1,\ c^a = c^{-1},\ c^b = c \right\rangle \end{aligned}$$

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The smallest admissible values, $n = 4$ and $m = 3$, correspond to the groups identified in the library of small groups of GAP [GAP19, BEO19] as [512, 456] and [512, 453], respectively.

Theorem. *The groups G and H are non-isomorphic but if k is a field of characteristic 2 then the group algebras kG and kH are isomorphic.*

Proof. To verify that G and H are non-isomorphic it suffices to observe that $C_G(G') = \langle z, x^2, xy \rangle$ and it has exponent 2^n while $C_H(H') = \langle c, a^2, b \rangle$ which has exponent 2^{n-1}.

If F is the prime field of k then $kG \cong k \otimes_F FG$ and therefore to prove that $kG \cong kH$ we may assume without loss of generality that k is the field with 2 elements. We will prove that kG and kH are isomorphic by identifying in kH a group basis isomorphic to G. Set

$$\widetilde{G} = \langle \widetilde{x}, \widetilde{y} \rangle \quad \text{where} \quad \widetilde{x} = a \quad \text{and} \quad \widetilde{y} = b(a + b + ab)c.$$

We will verify that $\widetilde{G} \cong G$ and $\widetilde{G}$ is a basis of kH. We use the following notation:

$$A = a + 1, \quad B = b + 1, \quad C = c + 1 \quad \text{and} \quad Y = \widetilde{y} + 1.$$

Let I denote the augmentation ideal of kH. Note that this ideal is nilpotent. It follows from Jennings' Theorem [Seh78, Theorem III.1.22] that $G/\Phi(G)$ is isomorphic to I/I^2 as elementary abelian groups and that $A + I^2$ and $B + I^2$ form a basis of I/I^2. By [KP69, 4.1.3] the intersection of the maximal right ideals of I equals I^2. Observe that $C = b^{-1}a^{-1}(ba - ab) = b^{-1}a^{-1}(BA - AB) \in I^2$. Thus

$$0 \equiv B(1 + AB)C = \widetilde{y} + b(1 + AB) + (1 + AB)C \equiv \widetilde{y} + b = Y + B \mod I^2.$$

Therefore the additive subgroup of I/I^2 is generated by $A + I^2$ and $Y + I^2$ and hence by the Burnside Basis Theorem [KP69, Theorem 4.1.4] we have that A and Y generate I as a ring.

We now verify that $\widetilde{x}, \widetilde{y}$ and $\widetilde{z} = [\widetilde{y}, \widetilde{x}]$ satisfy the relations of x and y in the presentation of G given above. Of course $\widetilde{x}^{2^n} = 1$. Moreover $\widetilde{x}^2 = a^2$ is central in kH and hence $1 = [\widetilde{y}, \widetilde{x}^2] = \widetilde{z}\ \widetilde{z}^{\widetilde{x}}$, so that $\widetilde{z}^{\widetilde{x}} = \widetilde{z}^{-1}$. Moreover

$$\widetilde{y}^2 = b^4c^2 + a^2(b^2c + b^4c^2) + a^2b^2c(b + bc).$$

which is central in kH because a^2, b^4, c^2 and b^2c belong to the center of H and the conjugacy class of b in H is $\{b, bc\}$. Then, as above we have $\widetilde{z}^{\widetilde{y}} = \widetilde{z}^{-1}$. Moreover,

$$\widetilde{y}^{2^m} = b^{2^{m+1}}c^{2^m} + a^{2^m}(b^{2^m}c^{2^{m-1}} + b^{2^{m+1}}c^{2^m}) + a^{2^m}b^{2^m}c^{2^{m-1}}(b^{2^{m-1}} + b^{2^{m-1}}c^{2^{m-1}}) = 1.$$

Finally, let J be the ideal of kH generated by C. Then kH/J is commutative and, as $\widetilde{z}$ is a commutator in the unit group of kH it follows that $\widetilde{z} \in 1 + J$. Then $1 + \widetilde{z} \in J$ and as $J^4 = 0$ it follows that $\widetilde{z}^4 = 1$.

We conclude that $\widetilde{G}$ is an epimorphic image of G and $\widetilde{G}$ includes a basis of kH as a k-vector space. As $|H| = |G|$ it follows that $\widetilde{G} \cong G$. ☐

The study of the groups from the theorem was inspired by the classification of 2-generated finite p-groups with cyclic derived subgroup [BGdR21]. Moreover, the LAGUNA package of GAP [BKRS19] was used for a first proof.

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