On the indecomposability of unit groups

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1 Introduction and statement of results

We consider units of orders in a simple algebra A of finite dimension over the rational field. Such A can be written as $A = M_n(D)$ where D is a skewfield (say of index d). Let K denote the centre of A (and D), R the integral closure of K, Λ an R-order in A, $\Gamma = \Lambda^\times$ the group of units of Λ and $\mathrm{S}\Gamma$ the kernel of the reduced norm map $\mathrm{nr} : A \to K$ on Γ. Two groups G and H are said to be commensurable (denoted $G \sim H$) if they have a common subgroup of finite index. Since $\Gamma \sim R^\times \times \mathrm{S}\Gamma$, the difficulty of Γ is concentrated in $\mathrm{S}\Gamma$. It is well known that $\mathrm{S}\Gamma \sim 1$ if and only if $A = K$ or $A = D$ is a totally definite quaternion algebra.

In the search for a structure theorem it is natural to envisage group theoretical constructions which produce "large" groups out of "small" ones. The most common examples are direct products, semidirect products, free products, amalgams and HNN extensions. We show that these constructions apply to unit groups only in very few cases of small dimension, the majority of which are well-known already. This demonstrates how far away we still are from an understanding of general unit groups.

Theorem 1 *If A is not commutative or a totally definite quaternion algebra then* $\mathrm{S}\Gamma$ *is virtually indecomposable as a direct product, that is: If* $\mathrm{S}\Gamma \sim H_1 \times H_2$ *then either* H_1 *or* H_2 *is finite.*

In particular, no torsionfree subgroup of finite index (e.g. a congruence subgroup) is indecomposable as a direct product.

Theorem 2 *If the group* $\mathrm{S}\Gamma$ *is commensurable with a non trivial free product, then one of the following conditions holds:*

1. *A is a quaternion algebra ramified at all but one of the infinite places of K or*
2. *$K = \mathbb{Q}$, $\dim_{\mathbb{Q}}(A) = 16$ and A is ramified at the infinite place.*

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In fact the only known (and presumably the only existing) case of a free product is when $A = M_2(\mathbb{Q})$, where $\mathrm{S}\Gamma$ is virtually free. A closer discussion of the other possible exceptions will be given in the course of the proof.

Amalgams (in particular free products) and HNN extensions are particular cases of tree products, that is of groups that act on trees (see [8]). Following Serre [8] we say that a group G has property (FA) if for every tree T on which G acts there is a fixed vertex, that is a vertex v of T such that v is fixed by all elements of G. One shows that a group has property (FA) if and only if it does not possess a non trivial decomposition as a tree product.

Theorem 3 *If* $\mathrm{S}\Gamma$ *is commensurable with a group* G *that acts on a tree without fixed vertices (that is* G *does not have property (FA), in particular, this happens if* G *is a non trivial amalgam or an HNN extension), then* $n \leq 2$ *and one of the following conditions holds:*

1. $nd = 2$ *and at least one infinite place is not ramified.*
2. $nd = 4$ *and at least one infinite place is ramified.*

The only known examples satisfying the assumptions of Theorem 3 are surface groups (e.g. if $A = D$ is a non-definite quaternion skewfield over $\mathbb{Q}$; see Zieschang [9] for decompositions) and some Bianchi groups (see Fine [2] for a rather complete discussion). Both correspond to case 1. No example of case of 2 seems to have been studied.

2 Proofs

Essentially, our theorems are applications of results on discrete subgroups of Lie groups. We set the stage. The real algebra $A \otimes_{\mathbb{Q}} \mathbb{R}$ can be written as

$$A \otimes_{\mathbb{Q}} \mathbb{R} = M_k(\mathbb{R})^{r_1} \times M_{k/2}(\mathbb{H})^{r_2} \times M_k(\mathbb{C})^s$$

where $k = nd$ is the degree of A, $r = r_1 + r_2$ is the number of real embeddings of K and $\dim_{\mathbb{Q}} K = r + 2s$. Our group $\mathrm{S}\Gamma$ is a lattice (a discrete subgroup of finite covolume) of the real Lie group H which is defined by the equation $\mathrm{nr}(x) = 1$, that is

$$H = A^1(\mathbb{R}) = \mathrm{SL}_k(\mathbb{R})^{r_1} \times \mathrm{SL}_{k/2}(\mathbb{H})^{r_2} \times \mathrm{SL}_k(\mathbb{C})^s.$$

(For more explanations see [5], where the notation is slightly different.) Furthermore, $\mathrm{S}\Gamma$ satisfies a certain condition of irreducibility which we do not define here (see [7]) but which follows in our case from the fact that the projections of $\mathrm{S}\Gamma$ to the factors of H are injective. The real rank of H (in the sense of Lie Theory) equals

$$\mathrm{rk}(H) = (r_1 + s)(k-1) + r_2(\frac{k}{2} - 1), \tag{1}$$

because for $\mathbb{K} \in \{\mathbb{R}, \mathbb{H} = \mathbb{H}(\mathbb{R}), \mathbb{C}\}$, $\mathrm{rk}(\mathrm{SL}_m(\mathbb{K})) = m - 1$. A Theorem due to Margulis, asserts that if $\mathrm{rk}(H) \geq 2$, then every group commensurable with $\mathrm{S}\Gamma$ satisfies the alternative (see [7, page 3]):

(MA) Every normal subgroup is either central or of finite index.

We also know from [6, Theorem 1], that $\mathbb{Q}[\Gamma] = A$ and therefore the centre of $\mathrm{S}\Gamma$ equals $\mathrm{S}\Gamma \cap K$ which is finite.

To prove Theorem 2 note that the exceptional cases correspond to the condition $\mathrm{rk}(H) < 2$ (see below). So we have to prove that if $\mathrm{rk}(H) \geq 2$ then $\mathrm{S}\Gamma$ is not commensurable with a non trivial free product. Passing to a torsionfree subgroup of finite index and using the Kurosh Subgroup Theorem we may assume that both factors are infinite. But then the normal closure of one factor is a noncentral subgroup of infinite index, contradicting (MA).

Before going ahead with the remaining proofs let us have a closer look at the cases excluded in Theorem 2. These correspond to the cases $\mathrm{rk}(H) < 2$ which are read off from equation (1) and split into the following cases:

(0) $\mathrm{rk}(H) = 0$.

  (a) $k = 1$, so that A is commutative.

  (b) $k = 2$ and $r_1 = s = 0$, so that A is a totally definite quaternion algebra.

(1) $\mathrm{rk}(H) = 1$ and $k = 2$.

  (a) $r_1 = 1$, $s = 0$, $n = 2$, $d = 1$;

  (b) $r_1 = 1$, $s = 0$, $n = 1$, $d = 2$;

  (c) $r_1 = 0$, $s = 1$, $n = 2$, $d = 1$;

  (d) $r_1 = 0$, $s = 1$, $n = 1$, $d = 2$;

(2) $\mathrm{rk}(H) = 1$ and $k = 4$.

  (a) $r_1 = s = 0$, $r_2 = 1$ and $n = d = 2$ and

  (b) $r_1 = s = 0$, $r_2 = 1$ and $n = 1$ and $d = 4$.

In case (0) $\mathrm{S}\Gamma$ is virtually abelian.

In cases (1a,c), $r_2 = 0$. So in case (1a) $A = M_2(\mathbb{Q})$ and $\mathrm{S}\Gamma$ is virtually free (and hence commensurable to a non trivial free product of cyclic groups). In fact this is the only case of a virtually free $\mathrm{S}\Gamma$ (see [5]). Case (1c) correspond to the Bianchi groups which most likely are not free products. This case has been extensively studied (see [2]).

In cases (1b,d) r_2 is arbitrary. Using a folklore lemma (stated below) we see that in (1b), $\mathrm{S}\Gamma$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$ (cocompact in (1b), by Hey's Theorem), in particular a Fuchsian group. It is well known that a torsionfree Fuchsian group is either free or a surface group. The surface groups does not decompose as free products because by a result of Hoare, Karras and Solitar [4] (see also [2, page 38]) the factors would be free products of cyclic groups and therefore the group will be virtually free which is impossible. In cases (1d), $\mathrm{S}\Gamma$ is a virtually a lattice in $\mathrm{SL}_2(\mathbb{C})$ (cocompact in (1d)), in particular a Kleinian group. This case has been studied, from a different point of view, in [1] but it seems not clear whether these groups can decompose as free products.

In case (2) we have $K = \mathbb{Q}$ and no example seems to have been worked out in detail.

Here is the folklore lemma used above:

Lemma 1 *Let X and Y topological groups, Y compact and A a discrete subgroup of $X \times Y$. Then the image of A in X is discrete.*

The proof is left to the reader. In our application Y is a product of copies of $\mathrm{SL}_1(\mathbb{H})$ (which is a 3-sphere).

To prove Theorem 3 we use a general theorem asserting that a factor of H having rank at least 2 enjoy Kazhdan's property (T) (see [3] for a thorough discussion of property (T)). So if *all* factors have rank at least 2, H as well as all lattices in it have property (T). Now a group having property (T) also has property (FA) [7, page 124].

To finish the proof of Theorem 3 it remains to check the possible exceptions. The only Lie groups not having (T) occurring as factors of H are $\mathrm{SL}_2(\mathbb{K})$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ (see [7, page 131] and note that $\mathrm{SL}_2(\mathbb{H})$ is locally isomorphic with $\mathrm{SO}(5,1)$). The factors $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$ occur if $dn \leq 2$ and $r_1 + s \neq 0$ (that is at least one infinite place is not ramified). The factor $\mathrm{SL}_2(\mathbb{H})$ occurs if $nd = 4$, and $r_2 \neq 0$ (that is one of the infinite places is ramified). In the first case $n \leq 2$ obviously and in the second it is a consequence of the fact that one infinite place is ramified.

Now we prove Theorem 1. If $\mathrm{rk}(H) \geq 2$ then the theorem is a direct consequence of (MA). Note that this argument also shows that $\mathrm{S}\Gamma$ is not a semidirect product of infinite groups except for the exceptional cases of Theorem 2. So it remains to prove the theorem for cases (0)-(2) above. Case (0) is excluded in the hypothesis. Assume that $\mathrm{S}\Gamma \sim H_1 \times H_2$ with both factors infinite. In case (1) it follows that $K(H_1)$ and $K(H_2)$ are two different subalgebras properly containing K, but commuting with each other elementwise, which is impossible in a quaternion algebra.

Case (2) requires more work. (Our argument only uses hypothesis $K = \mathbb{Q}$, thereby provides an alternative proof for some of the cases settled by (MA)). By [6] we have that $A = \mathbb{Q}(\mathrm{S}\Gamma) = A_1 A_2$ with $A_i = \mathbb{Q}(H_i)$ $(i = 1, 2)$. Thus the canonical map $A_1 \otimes A_2 \to A$ is onto. We next claim that A_1 is simple. In fact, a nilpotent two-sided ideal of A_1 would generate one of A, and a central idempotent of A_1 would be one of A. Now we can apply the Double Centralizer Theorem to obtain

$$\begin{aligned} \dim_{\mathbb{Q}} A &\leq \dim_{\mathbb{Q}} A_1 \dim_{\mathbb{Q}} A_2 \\ &\leq \dim_{\mathbb{Q}} A_1 \dim_{\mathbb{Q}} \mathrm{Cen}_A(A_1) = \dim_{\mathbb{Q}} A. \end{aligned}$$

and conclude that $A \simeq A_1 \otimes A_2$.

Let L_i be a maximal subfield of A_i $(i = 1, 2)$. Then $L_0 = L_1 \otimes L_2$ is a maximal subfield of A. Let $G_i = L_i \cap (H_1 \times H_2)$, $(i = 0, 1, 2)$. We claim that $G_0 = G_1 G_2$. The inclusion $G_1 G_2 \subset G_0$ is obvious. Assume that $g = h_1 h_2 \in G_0$ with $h_i \in H_i$. We must show that $h_i \in L_i$. This can be seen as follows: extending $\mathbb{Q}$ to a splitting field, we can think of the elements of A_i as matrices with entries in L_i, and of the products $a_1 a_2$ of the elements of A_i as being the Kronecker product of these matrices. But a nonzero Kronecker product is diagonal if and only if the factors are diagonal, which is exactly what is needed. (It is not difficult to give the argument a more rigorous form, by taking $\mathbb{Q}$-bases of L_i, L_i-bases of A_i, and noting that the products of these basis elements form a $\mathbb{Q}$-base of A.)

Now write $r(G)$ for the torsionfree rank of a finitely generated abelian group G and $r(F) = r(O^\times)$ for O and order in a number field F. Then $r(L_i) = r(G_i)$ because the reduced norm restricted to L_i is a power of the field norm. On the other hand from $G_0 = G_1G_2$ and the fact that $G_1 \cap G_2$ is finite we have that $r(G_0) = r(G_1) + r(G_2)$ and hence

$$r(L_0) = r(L_1) + r(L_2). \tag{2}$$

Now we show that this equality is impossible in our context. Denote by r_i and s_i the number of real and pairs of conjugate complex embeddings of L_i ($i = 0, 1, 2$). Then $L_0 = L_1 \otimes L_2$ and $K = \mathbb{Q}$ imply

$$r_0 = r_1r_2 \text{ and } s_0 = r_1s_2 + r_2s_1 + 2s_1s_2$$

By the Dirichlet Unit Theorem ($r(L_i) = r_i + s_i - 1$) and equation (2) we have

$$r_1r_2 + r_1s_2 + r_2s_1 + 2s_1s_2 + 1 = r_1 + s_1 + r_2 + s_2.$$

Setting $B_i = r_i + s_i \geq 1$, we obtain

$$B_1 + B_2 = B_1B_2 + s_1s_2 + 1 \geq B_1B_2 + 1$$

whence $(B_1-1)(B_2-1) \leq 0$. We conclude (say) $B_1 = 1$, that is L_1 is either $\mathbb{Q}$ or an imaginary quadratic field. Since L_1 was an arbitrary maximal subfield of A_1 we conclude that A_1 is either $\mathbb{Q}$ or a totally definite quaternion algebra. In both cases H_1 is finite, contradicting the hypothesis.

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