Approximating rings with local units via automorphisms. *

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Abstract

For a ring A with local units we investigate unital overrings T of A, and compare the automorphism groups $\mathrm{Aut}(A)$ and $\mathrm{Aut}(T)$.

1 Introduction, notation and preliminaries

We recall (see e.g. [4]) that a ring A is said to have *local units* provided for every finite subset $\mathcal{F}$ of A, there is an idempotent $e \in A$ such that $\mathcal{F} \subset eAe$. In particular, any unital ring has local units. In many ways, rings with local units behave very similarly to rings with identity (see, for example, [3], [4], [5], [6], and [9].) Therefore, it is natural to wonder if the essence of rings with local units is inherently connected with that of rings with units. More concretely, it is natural to attempt to "approximate" a ring with local units by a ring with identity. To do this, we must specify what we mean by an "approximation" of a ring with local units. For example, we expect the overring $\mathrm{End}(_AA)$ to be an acceptable approximation since the structure of A determines the endomorphisms of A and since A is dense in $\mathrm{End}(_AA)$. On the other hand, the classical extension $C(A)$ of A by the integers is not an acceptable approximation since A can have a non-zero annihilator in $C(A)$.

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We define an approximation of a ring with local units A to be an overring T of A with two mild conditions which makes A dense in T (see Section 2). We note that the class of approximations of A is closed under intermediate rings, and collapses to A itself if A has an identity.

With an appropriate definition of "approximation" in place, we want to relate properties of a ring with local units with properties of a "good unital approximation". In particular, for this paper, we are interested in the group of automorphisms of a ring A with local units. Along these lines, if A is a ring with local units and T is a unital approximation of A, we say that T has the **$\mathrm{Aut}(A)$-property** if $\mathrm{Aut}(T) \cong \mathrm{Aut}(A)$ via the restriction homomorphism. (We will often abuse the notation and write $\mathrm{Aut}(A) = \mathrm{Aut}(T)$ in this situation.) In this paper, we ask: Given a ring with local units A, is there an associated unital overring $T(A)$ such that $\mathrm{Aut}(A) \cong \mathrm{Aut}(T(A))$ via the restriction homomorphism? Our first result is of a negative nature:

Theorem A. (Theorem 3.1) There exists a ring A with local units such that no unital approximation satisfies the $\mathrm{Aut}(A)$-property.

Consequently, we conclude that, in general, there is no "good" unital approximation for a ring A satisfying the $\mathrm{Aut}(A)$-property.

Despite this result, there are a number of "classical" overring constructions that are useful for approximating $\mathrm{Aut}(A)$ when A has a specific structure. For example, A can be embedded inside the classical unitary extension $C(A)$ of A by the integers, or in the ring of left A-endomorphisms $E(A) = \mathrm{End}({}_A A)$, or in the multiplier ring (also known as the translational hull; see [10]), which we denote by $Q(A)$; see Definition 2.1. For each of these overrings T, every automorphism of A extends uniquely to an automorphism of T so that there is an injective group homomorphism from $\mathrm{Aut}(A)$ into $\mathrm{Aut}(T)$. Moreover, each of these overrings can fail to have the $\mathrm{Aut}(A)$-property; that is, the above-mentioned homomorphism need not be surjective.

With this unfortunate lack of complete agreement between full automorphism groups, we search for an overring T that approximates $\mathrm{Inn}(A)$, the group of inner automorphisms of A, or that approximates $\mathrm{Out}(A)$, the factor group $\mathrm{Aut}(A)/\mathrm{Inn}(A)$. Both of these groups are vital to the study of Picard groups; see, for example, [2] and [6]. It is well known that for a unital ring R, there is a group homomorphism $\Psi_R : \mathrm{Aut}(R) \to \mathrm{Pic}(R)$, where $\mathrm{Pic}(R)$ is the multiplicative group of isomorphism classes of invertible bimodules. In [6], the authors study the analogous map in the setting of rings with local units, and define the group $\mathrm{Inn}(A)$ of inner automorphisms of A to be the kernel of Ψ_A. If A is embedded as a two-sided ideal in a unital ring T, then the restriction to A of every inner automorphism of T is indeed in $\mathrm{Inn}(A)$.

If every automorphism of A extends to an automorphism of T so that there is a group homomorphism from $\mathrm{Aut}(A)$ to $\mathrm{Aut}(T)$, and if this group homomorphism maps the kernel of Ψ_A into the kernel of Ψ_T, then we can relate $\mathrm{Out}(A)$ to $\mathrm{Out}(T)$. This motivates the following definitions. We say that an overring T of A has the $\mathrm{Inn}(A)$**-property** (or the $\mathrm{Out}(A)$**-property**, respectively) if $\mathrm{Inn}(T) \cong \mathrm{Inn}(A)$ (or $\mathrm{Out}(T) \cong \mathrm{Out}(A)$, respectively) via the restriction homomorphism. We show that each of these classical overrings, $C(A), E(A)$ and $Q(A)$, can fail to have the $\mathrm{Out}(A)$-property and that both $C(A)$ and $E(A)$ can fail to have the $\mathrm{Inn}(A)$-property. However, we prove

Theorem B. (Corollary 4.5) $Q(A)$ has the $\mathrm{Inn}(A)$-property.

This result indicates that $Q(A)$ has a slight edge as an approximation to A over the other overring constructions.

Theorem B also shows that there are injective group homomorphisms from $\mathrm{Out}(A)$ into $\mathrm{Out}(T)$ for $T = Q(A)$ or $T = E(A)$ and so this makes $E(A)$ and $Q(A)$ candidates to satisfy the $\mathrm{Out}(A)$-property. Since the $\mathrm{Out}(A)$-property makes little sense unless automorphisms of A lift to automorphisms of the overring T, we consider "out-approximation" overrings (see Definition 5.1) and, similar to Theorem A, we prove the following result:

Theorem C. (Theorem 5.2) There exists a ring with local units A such that no out-approximation overring has the $\mathrm{Out}(A)$-property.

Despite this result, there are many rings A with local units such that $Q(A)$ or $E(A)$ have the $\mathrm{Out}(A)$-property. We provide examples of these as well.

Lest the reader lose any hope of approximating $\mathrm{Aut}(A)$ by $\mathrm{Aut}(T)$ for a unital overring T, we also show that for a large class of rings with local units, the multiplier ring satisfies all three conditions. That is, we prove:

Theorem D. (Theorem 6.2) Let P be a preordered set such that for every $i \in P$, $\{j \in P : i \leq j \text{ or } j \leq i\}$ is not finite. Let R be a unital ring and let A be the incidence ring of P over R (see Theorem 6.2 for definitions). Then $Q(A)$ has the $\mathrm{Aut}(A)$-, $\mathrm{Inn}(A)$-, and the $\mathrm{Out}(A)$-properties.

For example, this theorem easily applies to full and triangular matrix rings. In addition, we can also show that the other two overrings $C(A)$ and $E(A)$ fail to satisfy any of the $\mathrm{Aut}(A)$-, $\mathrm{Inn}(A)$-, or $\mathrm{Out}(A)$-properties.

The paper is organized as follows. In Section 2, we define an *approximation* of a ring with local units, and we show when the "classical" overrings are approximations. In Section 3, we prove Theorem A. We also consider several other examples of overrings which fail to have the $\mathrm{Aut}(A)$-property, notably $C(A)$ and the overring generated by A and the identity element of $E(A)$. Section 4 is devoted to the $\mathrm{Inn}(A)$-property and we prove Theorem B. In Section 5 we consider the $\mathrm{Out}(A)$-property, we prove Theorem C, and we provide examples in which $E(A)$ has the $\mathrm{Out}(A)$-property but $Q(A)$ does not, and vice-versa. Finally, we consider incidence rings in Section 6 and we prove Theorem D, namely, that $Q(A)$ has the $\mathrm{Aut}(A)$-, $\mathrm{Inn}(A)$-, and $\mathrm{Out}(A)$-properties while $E(A)$ and $C(A)$ need not.

2 Approximation overrings.

In this section, we define and discuss what is meant by an *approximation* of a ring with local units. We describe a number of such rings, including the three classical overrings mentioned in the introduction.

Definition 2.1 Let A be a fixed ring with local units. By an **overring** we mean a ring T such that $A \subset T$ and we say that an overring T of A is an **approximation overring** (or **an approximation of** A) if T satisfies

1. **The Function property:** $ATA = A$
2. **The Density property:** For every $0 \neq t \in T$, there exists $a, b \in A$ such that $atb \neq 0$.

Let $\mathcal{S}(A)$ denote the class of all ring isomorphism classes of approximation overrings of A. Denote the isomorphism class of $T \in \mathcal{S}(A)$ by $[T]$.

Next we present five examples of unital overrings as discussed in the introduction; we subsequently discuss situations in which these are approximation overrings.

Examples 2.2 Let A be a ring with local units.

1. **The unitary extension of A by the integers.** Let $C(A)$ be the set $\mathbf{Z} \times A$ and endow it with coordinate addition and with the multiplication given by $(m, a)(n, b) := (mn, mb + na + ab)$ for all $m, n \in \mathbf{Z}$ and $a, b \in A$. Since A embeds in $C(A)$ via $a \mapsto (0, a)$ and since $(1, 0)$ is the identity, $C(A)$ is a unitary overring of A.
2. **The endomorphism ring of A as a left module.** Let $E(A) = \text{End}({}_A A)$. This is an abuse of notation (since the ring $E(A)$ assumes left-sided homomorphisms) that we use to make the presentation easier. Clearly, $E(A)$ is a unitary overring of A via the embedding $a \mapsto \rho_a$ where ρ_a is the endomorphism of A given by right multiplication by a.

3. **The multiplier ring of A.** Let $Q(A)$ be the set

$$Q(A) = \{(f, g) \in \text{End}({}_AA) \times \text{End}(A_A) \mid \quad \text{for all} \quad a, b \in A, (af)b = a(gb)\}$$

with component-wise addition and multiplication. We leave to the reader the details of checking that this set satisfies the unital ring axioms and that A embeds in $Q(A)$ via the map $a \mapsto (\rho_a, \lambda_a)$. This construction also appears in the literature as a *multiplier* or *translational hull* (see [10]).

4. **The Burgess-Stewart overring of A.** For any ring R with identity, let $\kappa(R)$ denote the characteristic ring of R; this is the maximal epic (categorically speaking) extension of the image of the natural homomorphism from **Z** into R. Set $BS(A) = A + \kappa(\text{End}({}_AA)) \subset \text{End}({}_AA)$. See [7] for further details.

5. **The smallest overring containing A and $1_{Q(A)}$.** Let $O(A)$ denote the ring generated by $1_{Q(A)}$ and A; this unital overring is clearly contained in $Q(A)$.

We will soon motivate and justify the terminology used in our definition of approximation overrings but first we present some important observations.

Proposition 2.3 *Let A be a ring with local units. Then:*

1. *Every approximation overring T is isomorphic to a ring whose underlying set is a subset of $A^{A\times A}$; that is, there is a bijection between T and a subset of $A^{A\times A}$, the set of functions from $A \times A$ to A, and hence this subset can be endowed with a ring structure which makes it isomorphic to T.*
2. *Let $\alpha \in \text{Aut}(A)$, and let T be an approximation of A. If $\alpha' \in \text{Aut}(T)$ so that $\alpha'|_A = \alpha$, then α' is the unique extension of α to an automorphism of T.*
3. *The class of approximations of A is closed under intermediate subrings.*

4. *If A has an identity, then there is a unique approximation overring, namely A itself.*

5. *Every overring which contains A as a right (respectively, left) ideal is an approximation provided A has no annihilator on the right (respectively, left) in T. In this case, T can be viewed as a subring of* $\mathrm{End}({}_AA)$ *(respectively,* $\mathrm{End}(A_A)$*).*

6. *The overrings $E(A)$, $Q(A)$, $BS(A)$ and $O(A)$ are approximations of A.*

7. *The overring $C(A)$ is an approximation for A if and only if for each $n \in \mathbf{N}$ such that $nA \neq 0$, $nA \not\subset eA$ for all $e = e^2 \in A$.*

8. *If T is an approximation of A, then central elements of A are central in T.*

Proof. (1): Let T be a representative of an element of $\mathcal{S}(A)$. Since $ATA = A$, then we can define, for each $t \in T$, an A-bilinear function $\tau_t : A \times A \to A$ via $(a, b) \mapsto atb$. By the density condition, $\tau : t \mapsto \tau_t$ is an injection between T and a subset τ_T of $A^{A\times A}$. Thus, for each approximation overring T, there is a subset of $A^{A\times A}$ that can be endowed with a ring structure so that it becomes a ring isomorphic to T.

(2): Suppose $\beta \in \mathrm{Aut}(T)$ such that $\beta|_A = \alpha = \alpha'|_A$. Let $t \in T$ and we show that $\beta(t) = \alpha'(t)$. By the density property, it suffices to show that $x\beta(t)y = x\alpha'(t)y$ for all $x, y \in A$. But $x\beta(t)y = \beta(\beta^{-1}(x))\beta(t)\beta(\beta^{-1}(y)) = \beta(\beta^{-1}(x)t\beta^{-1}(y)) = \alpha'(\alpha'^{-1}(x)t\alpha'^{-1}(y)) = x\alpha'(t)y$.

(3) and (4): These are easy and left to the reader.

(5): Suppose A is a right ideal of an overring T. Clearly, $ATA = A$. For the density property, A is dense in $T \iff$ for each $t \in T$, there exists $a \in A$ such that $at \neq 0$. Thus, A is dense in T if and only if A has no non-0 right annihilators in T. The second statement is immediate.

(6): This follows from (3) and (5) above.

(7): First observe that $C(A)$ always has the function property so we need only check for the density property.

($\Rightarrow$): If $nA \subset eA$ for some $e = e^2$ in A, then for all $b \in A$, $nb = neb$ and so $(n, -ne)(0, b) = (0, nb - neb) = 0$. This is a contradiction to the density property.

($\Leftarrow$): Suppose $0 \neq (n, a) \in C(A)$ such that for all $c \in A$, $(n, a)(0, c) = (0, 0)$. Then for all $c \in A$, $nc + ac = 0$ and so if $e = e^2$ in A such that $ea = a$, then $nA \subset eA$, a contradiction. Hence, we have $(n, a)(0, c) \neq (0, 0)$ for some $c \in A$. The argument is finished with an application of (5) above.

(8): Let a belong to the center of A and let $t \in T$. By the density property of T, it suffices to show that $b(at - ta)c = 0$ for all $b, c \in A$. But using the function property of T and the commutativity of a, we have $batc = a(btc) = (btc)a = btac$ so that $batc - btac = 0$. ∎

Statement (1) of the above Lemma now justifies our decision to call an overring satisfying the function and density properties an approximation overring; these overrings truly reflect the structure of A since they can be viewed as rings contained in $A^{A \times A}$. Moreover, if T is an overring which represents an element of $\mathcal{S}(A)$, then the proof of 2.2(1) shows that the function property forces the multiplicative structure of T to reflect the multiplicative structure of A, while the density property ensures that T is in some sense "close" to A.

We observe that the construction of an overring with identity which is considered "best" by the authors of [7], namely $BS(A)$, is an approximation of A by (6) of the above Proposition.

Item (7) of Proposition 2.3 characterizes when $C(A)$ is an approximation overring. There are rings A with local units for which $C(A)$ is not an approximation overring as the next example shows.

Example 2.4 In general, $C(A)$ is not an overring that approximates A. Let $A = \mathbf{Z} \times (\oplus_{i \in \mathbf{N}} \mathbf{Z}_2)$. Then A is a ring with local units of characteristic 0 such that $2A \subset e_1 A$ where e_1 is the idempotent of A with 1 in the first coordinate and zeros elsewhere. It follows by Proposition 2.3

that $C(A)$ is not an approximation ring.

We complete this section with some comparisons between the multiplier ring $Q(A)$ and the endomorphism ring $E(A)$.

Remark 2.5 $Q(A)$ embeds in $E(A)$ via the ring homomorhism $(f,g) \mapsto f$. It suffices to show that this map is injective. Suppose $(0,g) \in Q(A)$. Then for $b \in A$, there is an idempotent $e \in A$ such that $gb = e(gb) = (e0)b = 0$. We identify $Q(A)$ with its image in $E(A)$. Under this association, it is easy to show that $Q(A)$ is the idealizer of A in $E(A)$.

The next result is well-known but we record it here for completeness.

Proposition 2.6 *Let $\sigma \in \mathrm{Aut}(A)$. Then σ extends uniquely to an automorphism of $Q(A)$ and to an automorphism of $E(A)$.*

Proof. We first extend σ to an automorphism of $E(A)$. For $f \in E(A)$, set $f^\sigma : {}_AA \to {}_AA$ where $(a)f^\sigma = (a^{\sigma^{-1}}f)^\sigma$. It is easy to check that $f^\sigma \in E(A)$, and that $\hat{\sigma} : E(A) \to E(A)$, defined by $\hat{\sigma}(f) = f^\sigma$, is an automorphism of $E(A)$ which extends σ. As $Q(A)$ is the idealizer of A in $E(A)$, it is easy to show that $\hat{\sigma}$ restricts to an automorphism of $Q(A)$.

Uniqueness is a consequence of Proposition 2.3. ∎

3 The $\mathrm{Aut}(A)$-property.

Our conclusion from the previous section is that the set $\mathcal{S}(A)$ is the "correct" set in which to look for overrings that approximate A and in particular $\mathrm{Aut}(A)$. Our first result in this section, however, shows that there is a ring with local units A such that no element of $\mathcal{S}(A)$ has the $\mathrm{Aut}(A)$-property. This is Theorem A from the introduction.

Theorem 3.1 *Let A' be any ring with local units and without an identity. The ring $A = A' \times \prod_{[T'] \in \mathcal{S}(A')} T'$ is a ring with local units such that no unital approximation of A has the* $\mathrm{Aut}(A)$-*property.*

Proof. Let C' denote $\prod_{[T'] \in \mathcal{S}(A')} T'$, so that $A = A' \times C'$. Note that by Proposition 2.3, $\mathcal{S}(A')$ is a set.

We claim that every approximation overring of A has the form $T' \times C'$ where T' is a unital approximation of A', and conversely, every overring of A of the form $T' \times C'$ where T' is a unital approximation of A' is an approximation overring.

To see this, let T be an approximation overring of A. Since A has the central idempotent $e = (0, 1_{C'})$ corresponding to the C'-component, T contains two central orthogonal idempotents, e and $1_T - e$, by Proposition 2.3. Moreover, $C' \subset eT$ and $A' \subset (1-e)T$ so that $T = (1-e)T \oplus eT$. Since $[T] \in \mathcal{S}(A)$, then $ATA = A$ and so $A'(1-e)TA' = A'$ and $C'eTC' = C'$. Since C' has an identity, the latter equation shows that $eT = C'$ so that T has the form $T = T' \oplus C'$, where $A'T'A' = A'$. We must show that $T' = (1_T - e)T$ has the density property. Let $t' \in T'$ so that $(t', 0) \in T$. Since A is dense in T, we can find $(a', c') \in A$ and $(b', d') \in A$ such that $0 \neq (a', c')(t', 0)(b', d') \in T$ and so $0 \neq a't'b' \in T'$, as desired.

For the converse statement, it suffices to show that overrings of the form $T' \times C'$ where $[T'] \in \mathcal{S}(A')$ satisfy the function and density properties. These are easily checked and left to the reader. This completes the claim.

Next let T' be any approximation overring of A' and write $T = T' \times C'$. We prove that there is an automorphism α_T of T that does not restrict to an automorphism of A. In this way, $\mathrm{Aut}(T) \neq \mathrm{Aut}(A)$. Write $T = T' \times C'$. Since $T' \in \mathcal{S}(A')$, there is a ring isomorphic copy T'' of T' that appears as a factor of C'. Let $\theta : T' \to T''$ be the ring isomorphism. Define α_T as follows: For $(t', c') \in T = T' \times C'$, write $c' = (t'', c'')$ where t'' belongs to T'' and c'' belongs to the product of the remaining overrings. Then set $\alpha_T(t', c') = (\theta^{-1}(t''), (\theta(t'), c''))$. It is straightforward to see that α_T is an automorphism of T. Moreover, an element of the form $(a', 0)$ gets mapped to an element of the form $(0, a'')$ so that A is not invariant under α_T. This completes the proof. ∎

Theorem 3.1 indicates that there are rings with local units such that no approximation overring has the Aut(A)-property. However, for certain types of rings with local units, there are such approximation overrings that we can easily construct. We now discuss various pros and cons of certain natural overrings, namely $Q(A)$ and $E(A)$, relative to the Aut(A)-property.

Example 3.2 An automorphism of $Q(A) = E(A)$ that does not leave A invariant. Let K be a ring with 1 and let R be the direct product of countably many copies of K. Let A be the direct sum of countably many copies of R and let Q be the direct product of countably many copies of R. It is straightforward to see that $Q \cong Q(A) = E(A)$.

We construct an automorphism α of Q which does not leave A invariant. Of course, we need some notation. For an element $c \in Q = \prod^\infty R$, we write $c = (c_i)$ and for an element $c_i \in R = \prod^\infty K$, we write $c_i = (c_{ij})$. Thus, $c = (c_i) = ((c_{ij}))$, which we will abbreviate as $c = (c_{ij})$. Define $\alpha \in \mathrm{Aut}(Q)$ via $(c)^\alpha = (c_{ij})^\alpha = (c_{ji})$. That is, α is the involution that interchanges the first and second subscript. It is an exercise left to the reader to show that α is an automorphism of Q. To see that A is not invariant under α, let 1_R denote the identity element of R and let $a = (1_R, 0, \ldots)$, which is an element of A. However, if $e_1 = (1, 0, \ldots) \in R$, then $a^\alpha = (e_1, e_1, e_1, \ldots)$ which is not in A.

Generally, $E(A)$ is worse than $Q(A)$ in terms of satisfying the Aut(A)-property because there are in fact inner automorphisms of E which do not restrict to automorphisms of A (so in particular they do not restrict to inner automorphisms of A).

Example 3.3 An example of a ring A such that $\mathrm{Aut}(A) = \mathrm{Aut}(Q)$ but $\mathrm{Aut}(A) \neq \mathrm{Aut}(E)$. Let $A = FM(R)$, the ring of matrices, indexed by the natural numbers, with only finitely many non-zero entries. Then $E(A)$ can be identified with $RFM(R)$, the ring of matrices indexed by the natural numbers such that every row has only finitely many non-zero entries and $Q(A)$ can be identified with $RCFM(R)$, the ring of matrices indexed by the natural numbers such that every row and every column has only finitely many non-zero entries. Using [1], we know that there is an inner automorphism of $E(A)$ that does not restrict to an automorphism of A so $\mathrm{Aut}(A)$ is properly contained in $\mathrm{Aut}(E(A))$. In particular, this shows that $E(A)$ satisfies neither the $\mathrm{Inn}(A)$-property nor the $\mathrm{Aut}(A)$-property. On the other hand, using either the results of [8, Proposition 3] or Theorem 6.2 we know that $\mathrm{Aut}(A) = \mathrm{Aut}(Q)$.

We close this section with a discussion of the unital overring $O(A)$. For the remainder of this section, we assume the notation of Example 3.2. We consider the subring $O(A)$ of Q formed by the almost constant sequences. That is, an element $c = (c_i) \in Q$ belongs to $O(A)$ if and only if there exists an N in $\mathbf{N}$ such that $c_n = c_N$ for every $n \geq N$. Note that if $K = \mathbf{Z}$, then $O(A)$ is precisely the overring generated by A and the identity element of $Q(A)$. Moreover, in this case, $O(A) \cong C(A) \cong BS(A)$. In this situation, we have the following positive result.

Proposition 3.4 *If* 0 *and* 1 *are the only idempotents of* K *and the characteristic of* K *is* 0*, then any automorphism of* $O(A)$ *restricts to an automorphism of* A.

Proof. Let α be an automorphism of $O(A)$ and assume that $\alpha(a) \notin A$ for some $a \in A$ Let $e_{i,j}$ denote the element of A which contains 1 in the jth coordinate of the ith coordinate and zeros everywhere else. By the assumption on K, the $e_{i,j}$ are precisely the primitive idempotents of Q and hence of A and $O(A)$. Therefore, α permutes these idempotents; in particular, there is a bijection $\sigma : \mathbf{N}^2 \to \mathbf{N}^2$ such that $\alpha(e_{i,j}) = e_{\sigma(i,j)}$. Let E_i denote the element of Q with the identity of R in the ith coordinate, and zero elsewhere. Then $e_{ij}E_i = e_{ij}$ and so $e_{\sigma(i,j)}\alpha(E_i) = \alpha(e_{i,j})\alpha(E_i) = \alpha(e_{i,j}) = e_{\sigma(i,j)}$. This implies that $\alpha(E_i)_{\sigma(i,j)} = 1$ for every $j \in \mathbf{N}$. On the other hand, if $i \neq k$, then $e_{k,j}E_i = 0$ and so $e_{\sigma(k,j)}\alpha(E_i) = \alpha(e_{k,j})\alpha(E_i) = 0$. Therefore, $\alpha(E_i)_{\sigma(k,j)} = 0$. Since $A = \cup_{n\in\mathbf{N}}(\sum_{i=1}^{n} E_i)O(A)$, then there exists at least one i such that $\alpha(E_i) \notin A$. This implies that there exist two sequences $(k_n)_{n\in\mathbf{N}}$ and $(l_n)_{n\in\mathbf{N}}$ of the natural numbers such that $(k_n)_{n\in\mathbf{N}}$ is strictly ascending and $\alpha(E_i)_{k_n,l_n} \neq 0$ for every $n \in \mathbf{N}$. By the hypothesis on K, $\alpha(E_i)_{k_n,l_n} = 1$. Since $\alpha(E_i) \in O(A)$ and the sequence k_n is strictly ascending, there exists some m such that $\alpha(E_i)_{k_n} = \alpha(E_i)_{k_m}$ for every $n \geq m$. Therefore, $\alpha(E_i)_{k_n,l_m} = \alpha(E_i)_{k_m,l_m}$ and so we may assume that the sequence $l_n = l_m$ for every $n \geq m$. Set $(i_n, j_n) = \sigma^{-1}(k_n, l_n)$. Then $\alpha(E_i)_{\sigma(i_n,j_n)} = 1$ for every n. Hence $i_n = i$ for every n, and $j_n \neq j_m$ for $m \neq n$. Let $r \in A$ be defined by $r_{k,j} = \delta_{ik}j$. Clearly, $e_{i,j}r = je_{i,j}$. Therefore, $e_{\sigma(i,j)}\alpha(r) = je_{\sigma(i,j)}$ and so $\alpha(r)_{\sigma(i,j)} = j$ for every j. This implies that $\alpha(r)_{k_n,l_n} = \alpha(r)_{\sigma(i,j_n)} = j_n$, and, hence, if $n \geq m$, $\alpha(r)_{k_n,l_m} = j_n$, Since all the j_n are different, and (k_n) is strictly ascending, then $\alpha(r) \notin O(A)$. ∎

It appears, after considering the above Proposition and Example 3.2, that this ring $O(A)$ is "not worse" than $Q(A)$ in terms of automorphisms. However, with slight modification of the coordinate ring, we see that $Q(A)$ is "not worse" than $O(A)$.

Example 3.5 Let T be the subring of $K^{\mathbf{N}}$ of almost constant sequences. Let A' be the direct sum of $\mathbf{N}$-copies of T. As before $Q' = Q(A')$ is the direct product of $\mathbf{N}$-copies of T and we set $O(A)$ to be as defined above; namely, the subset of Q' consisting of almost constant sequences over T. It is clear that the automorphism α restricts to an automorphism of $O(A)$ which does not preserve A'. But an argument similar to that found in Proposition 3.4 shows that if 0 and 1 are the only idempotents of K and the characteristic of K is 0, then every automorphism of Q' restricts to an automorphism of A'.

Our conclusions are that there are rings with local units A for which $\mathrm{Aut}(A)$ cannot be approximated by (approximating) unital overrings. However, in Section 4, we will see that the multiplier ring $Q(A)$ will always approximate $\mathrm{Inn}(A)$ and for a nice class of rings with local units, $Q(A)$ also approximates $\mathrm{Aut}(A)$ and $\mathrm{Out}(A)$. The other approximating overrings such as $E(A)$, $O(A)$ or $C(A)$ do not fare as well.

4 The $\mathrm{Inn}(A)$-property.

In this section, we show that $Q(A)$ satisfies the $\mathrm{Inn}(A)$-property. Of course, since A need not have an identity, it is important to understand what is meant by an inner automorphism of A. In [6], the authors characterize the elements of the kernel of the map from $\mathrm{Aut}(A)$ to $\mathrm{Pic}(A)$ as those which are inner in the following sense.

Definition 4.1 Let $\phi \in \mathrm{Aut}(A)$ and denote the set of idempotents of A by $\tilde{E}$. We say ϕ is *inner* provided there are two maps $u, v : \tilde{E} \to A$ such that for every $x \in eAf$, $e, f \in \tilde{E}$, $\phi(x) = u(e)xv(f)$.

If A is a 2-sided ideal in a ring B with identity, then any inner automorphism of B restricts to an inner automorphism of A; see [6, Remark 1.2].

Remark 4.2 Assume that $\alpha \in \mathrm{Aut}(A)$ is inner and let u, v be the maps in the definition. Let $e \in \tilde{E}$.

1. We may assume that $u(e) \in \alpha(e)Ae$ and $v(e) \in eA\alpha(e)$ because

$$\alpha(x) = (\alpha(e)u(e)e)x(ev(e)\alpha(e))$$

   and hence $u(e)$ and $v(e)$ could be substituted by $\alpha(e)u(e)e$ and $ev(e)\alpha(e)$. From now on, we assume that u and v always satisfies this condition.

2. $\alpha(e) = u(e)v(e)$.

3. If $x \in eAe$ then

$$\begin{aligned} x = eAe &= \alpha(\alpha^{-1}(e))x\alpha(\alpha^{-1}(e)) \\ &= u(\alpha^{-1}(e))v(\alpha^{-1}(e))xu(\alpha^{-1}(e))v(\alpha^{-1}(e)) \\ &= \alpha(v(\alpha^{-1}(e))xu(\alpha^{-1}(e))). \end{aligned}$$

Thus, $\alpha^{-1}(x) = v(\alpha^{-1}(e))xu(\alpha^{-1}(e))$, $u'(e) = v(\alpha^{-1}(e)) \in \alpha^{-1}(e)Ae$ and $v'(e) = u(\alpha^{-1}(e)) \in \alpha^{-1}(e)Ae$. Consequently, u', v' satisfy the corresponding properties for α^{-1}. The same computation shows that if α is inner and u, v satisfy the corresponding properties for Definition 4.1, then so do u', v'.

4. $v(e)u(e) = e$. (This is a direct consequence of 2 and 3).

Lemma 4.3 (c.f., [6]) *Let $\alpha \in \mathrm{Aut}(A)$ and $u, v : \tilde{E} \to A$ such that, for every $e \in \tilde{E}$, $u(e) \in \alpha(e)Ae$ and $v(e) \in eA\alpha(e)$, and, for every $x \in eAe$, $\alpha(x) = u(e)xv(e)$. Then the following conditions are equivalent:*

1. *For every e, f idempotents of A and $x \in A$, $\alpha(x) = u(e)xv(f)$.*
2. *For every e, f idempotents of A with $eAe \subseteq fAf$, $u(f)e = u(e)$.*
3. *For every e, f idempotents of A with $eAe \subseteq fAf$, $ev(f) = v(e)$.*
4. *For every e, f orthogonal idempotents of A, $u(e + f) = u(e) + u(f)$.*
5. *For every e, f orthogonal idempotents of A, $v(e + f) = v(e) + v(f)$.*
6. *For every e, f idempotents of A with $eAe \subseteq fAf$, $\alpha(e)u(f) = u(e)$.*
7. *For every e, f idempotents of A with $eAe \subseteq fAf$, $v(f)\alpha(e) = v(e)$.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) has been proved in [6]. The equivalence (1) $\Leftrightarrow$ (4)$\Leftrightarrow$ (5)$\Leftrightarrow$ (6) $\Leftrightarrow$ (7) can be proved similarly. ∎

Theorem 4.4 *Let A be a ring with local units, and let B be any unital ring which contains $Q(A)$ as a unital subring; i.e., $1_{Q(A)} = 1_B$. Then every inner automorphism of A is the restriction of an inner automorphism of B. In particular, the extension of any $\alpha \in \mathrm{Inn}(A)$ to an automorphism $\alpha' \in \mathrm{Aut}(B)$ is an inner automorphism of B.*

Proof. It suffices to prove the result when $B = Q(A)$. For the purposes of this proof, we will write all maps on the left. Assume that $\alpha \in$Aut(A) is inner and let $u, v : \tilde{E} \to A$ maps satisfying Definition 4.1. We set u' and v' as in Remark 4.2.

First note that u and v can be extended to A as follows: Given $x \in A$, let $e, f \in \tilde{E}$ such that $x \in eAf$ then one defines

$$u(x) = u(e)x \quad \text{and} \quad v(x) = xv(f)$$

Note that u is well defined because if e_1 and e_2 are two idempotents so that $x \in e_1A \cap e_2A$ then there exist another idempotent e_3 such that $e_1Ae_1 \cap e_2Ae_2 \subseteq e_3Ae_3$ and hence as a consequence of Lemma 4.3

$$u(e_1)x = u(e_3)e_1x = u(e_3)x = u(e_3)e_2x = u(e_2)x.$$

Similarly $xv(f_1) = xv(f_2)$ if $x \in Af_1 \cap Af_2$. In an analogous way, one extends u' and v' to A. Obviously $u, u' \in$End(A_A) and $v, v' \in$End$(\,_AA)$. Furthermore, if $e_1, e_2 \in \tilde{E}$ there exists $f \in \tilde{E}$ so that $e_1, e_2, \alpha^{-1}(e_1), \alpha^{-1}(e_2) \in fAf$ and hence

$$e_1u(e_2) = e_1u(f)e_2 = \alpha(\alpha^{-1}(e_1))u(f)e_2 = u(\alpha^{-1}(e_1))e_2 = v'(e_1)e_2$$

and

$$e_1u'(e_2) = e_1v(\alpha^{-1}(e_2)) = e_1v(f)e_2 = v(e_1)e_2.$$

Therefore, $(v', u), (v, u') \in Q$. Furthermore, if $x \in eAe$ then $v(\alpha^{-1}(e)) \in \alpha^{-1}(e)Ae$ and $u(\alpha^{-1}(e)) \in eA\alpha^{-1}(e)$ and hence

$$(u \circ u')(x) = u(u'(e)x) = u(v(\alpha^{-1}(e))x) = u(\alpha^{-1}(e))v(\alpha^{-1}(e))x = ex = x$$

and

$$(v \circ v')(x) = v(xv'(e)) = v(xu(\alpha^{-1}(e))) = xu(\alpha^{-1}(e))v(\alpha^{-1}(e)) = xe = x.$$

Furthermore,

$$(u' \circ u)(x) = u'(u(e)x) = u'(\alpha(e))u(e)x = v(e)u(e)x = ex = x$$

and

$$(v' \circ v)(x) = v'(xv(e)) = xv(e)v'(\alpha(e)) = xv(e)u(e) = xe = x.$$

Therefore, (v', u) and (v, u') are inverses to each other. On the other hand if $e \in \tilde{E}$ and $x \in eAe$, then

$$(v', u)x(v, u') = v(u(x)) = u(e)xv(e) = \alpha(x). \blacksquare$$

Thus we have shown that $Q(A)$ is, in some sense, the "smallest" unital ring B to which inner automorphisms of A extend to inner automorphisms of B. But by [6, Remark 1.2], if A is a 2-sided ideal in a unital ring B, then any inner automorphism of B restricts to an inner automorphism of A. As $Q(A)$ is the idealizer of A in $E(A)$, $Q(A)$ is a ring of this type (in some sense, the "largest" ring of this type). Thus we have

Corollary 4.5 *$Q(A)$ has the* $\mathrm{Inn}(A)$*-property.*

We bring to the attention of the reader that the proof of Theorem 4.4 actually constructs the unit of $Q(A)$ that makes the automorphism inner; this unit comes from the original u and v.

We have seen that $E(A)$ does not have the $\mathrm{Inn}(A)$-property (see [1] or Example 3.3). However, we do have

Corollary 4.6 $\mathrm{Inn}(A) = \mathrm{Inn}(E(A)) \cap \mathrm{Aut}(A) \subset \mathrm{Inn}(E(A))$.

Proof. Theorem 4.4 gives that $\text{Inn}(A) \subset \text{Inn}(E(A))$. For the other containment, if $\alpha \in \text{Inn}(E(A)) \cap \text{Aut}(A)$, then there is a unit u of $E(A)$ so that $uAu^{-1} = A$. It follows that $u \in Q(A)$ and so $\alpha \in \text{Inn}(A)$ by [6, Remark 1.2]. ∎

Finally, we close this section with an example which indicates the overrings $O(A)$, $C(A)$, and $BS(A)$ do not in general have the $\text{Inn}(A)$-property.

Example 4.7 Let R be any unital ring of characteristic zero with a non-central unit u. Let A be a countably infinite direct sum of copies of R and let Q be the countably infinite direct product so that $Q = Q(A)$. Let $\hat{u}$ be the unit of Q that has u in each coordinate and let σ be the inner non-trivial automorphism by $\hat{u}$ of Q; by [6, Remark 1.2], σ restricts to an inner automorphism of A. Let $O(A)$ be the ring $\langle A, 1_Q \rangle$ generated by A and 1_Q. Since R has characteristic zero, $O(A) = C(A) = BS(A)$. If v is a unit of $O(A)$, then there exists an integer N such that, for every $j > N$, the jth coordinate of v is either 1 or -1. It follows that $\sigma|_A$ does not extend to an inner automorphism of $O(A)$.

5 The Out(A)-property.

We have seen that, in general, there is no unital approximation overring to a ring with local units with the $\text{Aut}(A)$-property, but that $Q(A)$ is always an approximation overring that satisfies the $\text{Inn}(A)$-property. We are left with considering the $\text{Out}(A)$-property; that is, the property that $\text{Out}(A) \cong \text{Out}(T)$ for some approximation overring T of A via the restriction homomorphism. We present a result that has the same flavor as Theorem A from the introduction; namely, that there is a ring with local units for which no approximation overring has the $\text{Out}(A)$-property. In order for this property to make sense, however, we will make the following assumption.

Definition 5.1 An approximation overring T of A is an **out-approximation** if every automorphism ϕ of A extends (uniquely) to an automorphism ϕ' of T and that if $\phi \in \mathrm{Inn}(A)$, then $\phi' \in \mathrm{Inn}(T)$. In this way, we have a unique group homomorphism $\mathrm{Out}(A) \to \mathrm{Out}(T)$ which makes the following diagram commutative:

$$\begin{array}{ccccccccc} 1 & \to & \mathrm{Inn}(A) & \to & \mathrm{Aut}(A) & \to & \mathrm{Out}(A) & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \mathrm{Inn}(T) & \to & \mathrm{Aut}(T) & \to & \mathrm{Out}(T) & \to & 1. \end{array}$$

By Proposition 2.6 and Theorem 4.4, it is straightforward to show that both $Q(A)$ and $E(A)$ are out-approximation overrings.

Theorem 5.2 *There exists a ring with local units A such that no out-approximation overring has the* $\mathrm{Out}(A)$*-property.*

Proof. Let A_1 be any ring with local units but without an identity. Let $\mathcal{T} = \{T_i \mid i \in I\}$ be a set of representatives (under ring isomorphism) of unital, out-approximation overrings of A_1. Let $T' = \prod_{i \in I} T_i$ and let $A = A_1 \times T'$.

Let T be any unital out-approximation overring of A. By the proof of Theorem 3.1, T has the form $T = S \times T'$ where S is an approximation overring of A_1.

We claim that S is an out-approximation overring of A_1. To this end, let $\psi \in \mathrm{Aut}(A_1)$ and let $\psi \times 1 \in \mathrm{Aut}(A)$, where 1 is the identity automorphism of T'. Since T is an out-approximation of A, $\psi \times 1$ extends to $\Psi \in \mathrm{Aut}(T)$. We first show that Ψ restricts to an automorphism of S. Let $s \in S$; it suffices to show that $(0,1)\Psi(s,0) = (0,0)$, where $(0,1)$ is a central idempotent of A corresponding to T'. By the density property for T, it suffices to show that $(0,1)\Psi(s,0)\Psi(a,t_1) = (0,0)$ for all $(a,t_1) \in A$. But $(0,1)\Psi(s,0)\Psi(a,t_1) = (0,1)\Psi(sa,0) = (0,1)(\psi(sa),0) = 0$. Thus, Ψ sends S into S and so $\Psi|_S$ is an automorphism ψ' of S and $\psi'|_{A_1} = \psi$. Moreover, we can write $\Psi = \psi' \times 1$.

To complete the claim, we must show that if $\psi \in \mathrm{Inn}(A_1)$, then $\psi' \in \mathrm{Inn}(S)$. However, if $\psi \in \mathrm{Inn}(A_1)$, then $\Psi \in \mathrm{Inn}(T)$. Since the units of T are products of the units of S and the units of T', we have $\psi' \in \mathrm{Inn}(S)$.

Now we show that T does not have the Out(A)-property. Let $\alpha_T \in \mathrm{Aut}(T)$ be the automorphism defined in the proof of Theorem 3.1. We show that the coset $\alpha_T\mathrm{Inn}(T) \neq \beta\mathrm{Inn}(T)$ for any automorphism β that is an extension of an automorphism of A. Suppose $\alpha_T = \beta\iota_u$ where b restricts to an automorphism of A and ι_u is conjugation by a unit $u \in T$. Without loss of generality, write $T' = S \times S'$ so that $T = S \times T' = S \times (S \times S')$. Since the units of T are tuples of units of the coordinate rings, we have $\beta(0 \times S \times 0) = \beta\iota_u(0 \times S \times 0) = \alpha_T(0 \times S \times 0) = (S \times 0 \times 0)$. But this shows that β does not restrict to an automorphism of A and so we have a contradiction. ∎

So, as with the Aut(A)-property, there does not exist an out-approximation overring construction that, in general, has the Out(A)-property. Nonetheless, as with the Aut(A)-property, there are out-approximation overrings that do satisfy the Out(A)-property, for specific classes of rings with local units. For example, if A is the ring of countably infinite square finite matrices over a ring R with identity, then both $Q(A)$ and $E(A)$ have the Out(A)-property; see [8]. We finish this section with one example that shows that $E(A)$ can have the Out(A) property while $Q(A)$ need not, and another example that shows that $Q(A)$ can have the Out(A) property while $E(A)$ need not. We begin with a remark.

Remark 5.3 The Aut(A)-property is equivalent to the Out(A)-property for every out-approximation overing T with the Inn(A)-property. Let T be an out approximation of A so that T also has the Inn(A)-property. Then every automorphism of T is a product of an inner automorphism of T and an automorphism of A. But by the Inn(A)-property, every inner automorphism of T is an automorphism of A so every automorphism of T is also an automorphism of A. The converse is clear.

In particular, this holds when $T = Q(A)$.

Lemma 5.4 *Let A be a ring with local units and let T be any overring of A such that A is a right ideal of T.*

1. *Let $\alpha \in \mathrm{Aut}(T)$ and suppose there exists a unit u of T and $\beta \in \mathrm{Aut}(T)$ such that $\alpha = \iota_u \beta$, where ι_u denotes conjugation by u and β is restricts to an automorphism of A. Then $A = A\alpha(A)$.*
2. *If every automorphism of A lifts to an automorphism of T and if T contains $Q(A)$, then T embeds inside $E(A)$ and there is a natural, injective map $\phi : \mathrm{Out}(A) \to \mathrm{Out}(T)$ via $\alpha\mathrm{Inn}(A) \mapsto \alpha'\mathrm{Inn}(T)$ (where α' is the extension of α).*

Proof. (1): As A is a right ideal, we always have that $A\alpha(A) \subset A$ so it suffices to show the opposite inclusion. Let $a \in A$. First assume that $\alpha = \iota_u$. Since A is a right ideal of T, $au \in A$ and so there exists $e = e^2$ such that $au = aue$. Thus, $a = aueu^{-1} = a\alpha(e)$ so $A = A\alpha(A)$.

Next suppose α restricts to an automorphism of A. Then $a = ea = e\alpha(\alpha^{-1}a)$ and so $a \in A\alpha(A)$. Hence, $A = A\alpha(A)$.

Finally, we claim that if γ and δ are automorphisms of T such that $A = A\gamma(A)$ and $A = A\delta(A)$, then $A = A\gamma\delta(A)$. But $A\gamma\delta(A) = (A\gamma(A))\gamma\delta(A) = A\gamma(A\delta(A)) = A\gamma(A) = A$.

(2): To see that T embeds inside $E(A)$, let $\rho : T \to E(A)$ via $t \mapsto \rho_t$ where ρ_t is right multiplication by t. Since A is a right ideal of T, $\rho_t \in E(A)$. The density property of T yields that ρ is injective.

If $\alpha \in \mathrm{Inn}(A)$, then α lifts to an automorphism α' of T and, by Theorem 4.4, α' is inner on T. Hence, the map ϕ is well-defined. Suppose $\alpha\mathrm{Inn}(A) \in \ker\phi$ so that $\alpha' \in \mathrm{Inn}(T)$. Thus, there is a unit $u \in T$ such that $\alpha'(x) = uxu^{-1}$ for all $x \in A$. Since α' is the extension of α, then $uxu^{-1} \in A$ for all $x \in A$. Hence, since A is a right ideal of T, $uA \subset A$. Viewing T inside $E(A)$, it follows that

$u \in Q(A)$. But $\alpha \in \mathrm{Inn}(Q) = \mathrm{Inn}(A)$ so that ϕ is injective. ∎

Now for our first example.

Example 5.5 An example of a ring A such that E has the $\mathrm{Out}(A)$-property but Q does not. Let k be a field, let $A_1 = FM(k)$, the finite matrices indexed by $\mathbf{N}$ with entries from k, let $Q_1 = Q(A_1)$, and let $E_1 = \mathrm{End}({}_{A_1}A_1)$. Note that $Q_1 \cong RCFM(k)$, the row and column finite matrices indexed by $\mathbf{N}$ with entries from k, and $E_1 = RFM(k)$ is the ring of row finite matrices indexed by $\mathbf{N}$ with entries from k.

Set $A = A_1 \times Q_1$ and notice that $Q = Q(A) = Q(A_1) \times Q_1 = Q_1 \times Q_1$ and $E = \mathrm{End}({}_AA) = E_1 \times Q_1$. The involution automorphism of Q that interchanges the coordinates $\alpha(x, y) = (y, x)$ does not restrict to an automorphism of A and so Q does not have the $\mathrm{Aut}(A)$-property and so by the Remark above, Q does not have the $\mathrm{Out}(A)$-property.

We show that E has the $\mathrm{Out}(A)$-property. Let $\alpha \in \mathrm{Out}(E)$. Let (e_{ij}) be the set of matrix units in A_1. Set $e^1_{ij} = (e_{ij}, 0)$ and $e^2_{ij} = (0, e_{ij})$. Let $\mathcal{E}_1 = \{e^1_{ij}\}$, $\mathcal{E}_2 = \{e^2_{ij}\}$ and $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$. We identify E_1 with $E_1 \times \{0\}$ and Q_2 with $\{0\} \times Q_1$. Let $e^1_1 = e^1_{11}$. Since this is a primitive idempotent, either $\alpha(e^1_1) \in E_1$ or $\alpha(e^1_1) \in Q_1$.

Case 1: $\alpha(e^1_1) \in E_1$.

In this case, we show that there exists $\alpha_1 \in \mathrm{Aut}(E_1)$ and $\alpha_2 \in \mathrm{Aut}(Q_1)$ so that $\alpha = \alpha_1 \times \alpha_2$.

Now for all i, j, k, l, $e^1_{kl} \in Ee^1_{ij}E = FC(k) \times \{0\}$ (here $FC(k)$ denotes the set of matrices with only finitely many non-zero columns) and $e^2_{k,l} \in Ee^2_{ij}E = \{0\} \times A$. Thus, $\alpha(e^1_{k,l}) \in E\alpha(e^1_{ij})E$ and $\alpha(e^2_{k,l}) \in E\alpha(e^2_{ij})E$. Consequently, it follows that $\alpha(e^1_{ij}) \in E_1$ and $\alpha(e^2_{ij}) \in Q_1$ for all i, j.

Next we prove that $\alpha(E_1) \subset E_1$ and $\alpha(Q_1) \subset Q_1$. If $a \in E_1$ and $\alpha(a) \notin E_1$, then there exists an i such that $\alpha(a)e^2_i \neq 0$, and hence $a\alpha^{-1}(e^2_i) \neq 0$. But this is a contradiction since $a \in E_1$ and $\alpha^{-1}(e^2_i) \in Q_1$. Hence, we have $\alpha(E_1) \subset E_1$ and similarly $\alpha(Q_1) \subset Q_1$.

Letting $\alpha_1 = \alpha|_{E_1}$ and $\alpha_2 = \alpha|_{Q_1}$, it follows that $\alpha = \alpha_1 \times \alpha_2$ such that $\alpha_1 \in \text{Aut}(E_1)$ and $\alpha_2 \in \text{Aut}(Q_1)$.

Case 2: $\alpha(e_1^1) \in Q_1$.

We show that this case cannot occur. Similar arguments to case 1 above shows that $\alpha(E_1) \subset Q_1$ and $\alpha(Q_1) \subset E_1$. Consequently, there exist isomorphisms $\alpha_2 : E_1 \to Q_1$ and $\alpha_1 : Q_1 \to E_1$ such that $\alpha(x, y) = (\alpha_1(y), \alpha_2(x))$. But this contradicts [8, Theorem 8].

Now by [8, Proposition 6], there exists $\sigma \in \text{Aut}(A_1)$ and $\tau \in \text{Inn}(E_1)$ such that $\alpha_1 = \sigma\tau$ and so $\alpha = (\sigma, \alpha_2)(\tau, 1) \in \text{Aut}(A)\text{Inn}(E)$. This proves that $\text{Aut}(E) = \text{Aut}(A)\text{Inn}(E)$ and so by Lemma 5.4, $\text{Out}(A) = \text{Out}(E)$.

Finally, we have an example of the "converse" situation to the previous example.

Example 5.6 An example of a ring A such that Q has the $\text{Out}(A)$-property but E does not.

Let k be a field, let $E_1 = LTM(k)$ denote the lower triangular matrices indexed by $\mathbf{N}$ with entries from k, and let $E = LTM(E_1)$. Let Q denote the subring of E consisting of those matrices such that every column has only finitely many non-zero entries; that is, $Q = CFLTM(E_1)$. Finally, let A be the subring of Q consisting of those matrices with only finitely many non-zero entries. Using appropriate identifications, we identify E with $E(A)$ and Q with $Q(A)$. By Theorem 6.2 below, $\text{Aut}(Q) = \text{Aut}(A)$; thus, $\text{Out}(Q) = \text{Out}(A)$ by Remark 5.3. We claim that $\text{Out}(A) \neq \text{Out}(E)$.

For $M \in E$, write $M = (M_{ij})$ and $M_{ij} = M_{ijkl}$ where $i, j, k, l \in \mathbf{N}$; thus, $M = (M_{ijkl})$. Define $\alpha : E \to E$ via $\alpha(M) = \hat{M}$ where $\hat{M}_{ijkl} = M_{klij}$. A tedious exercise shows that α is an automorphism of E. Further, α does not restrict to an automorphism of A. To see this, consider the element e_{11} which contains the identity of E_1 in the (1,1)-entry and zeros elsewhere. Since the identity of E_1 is the identity matrix of $LTM(k)$, $\alpha(e_{11}) \notin A$.

To show that $\mathrm{Out}(E) \neq \mathrm{Out}(A)$, it suffices, by Lemma 5.4, to show that $A \neq A\alpha(A)$. Let $e = e_{11}$ be the matrix above. Suppose $e = ab$ where $a \in A$ and $b = \alpha(c)$ where $c \in A$. Let a_{11} be the (1,1)-entry of a and let b_{11} be the (1,1)-entry of b. Thus, $a_{11}b_{11} = 1_{E_1}$, the identity of E_1. But because $b = \alpha(c)$ where $c \in A$, b_{11} has finitely many non-zero columns and hence so does the product $a_{11}b_{11}$. Since 1_{E_1} has infinitely many non-zero columns, we have a contradiction.

6 Incidence rings.

In this section, our aim is to show that the multiplier ring $Q(A)$ is a good approximation for a variety of rings with local units, including incidence rings generated by infinite posets that have a mild finiteness condition.

Definition 6.1 Let R be any ring with identity, let $(P, \leq)$ be any (possibly infinite) partially ordered set ("poset"), and let $A(P, R)$ denote the incidence ring. $A(P, R)$ may be viewed as the collection of $P \times P$ matrices (i.e., matrices that are indexed by P) with entries from R, having at most finitely many nonzero entries, for which the entry in the x-row, y-column is zero whenever $x \not\leq y$ $(x, y \in P)$.

For each $x \in X$ we let f_x denote the element of $A(P, R)$ which is 1_R in the (x, x) coordinate, and zero elsewhere. $A(P, R)$ has an identity if and only if P is finite. If P is infinite, then the collection

$$F = \{\sum_{x \in L} f_x \mid L \text{ is a finite subset of } X\}$$

is a set of local units for $A(P, R)$.

We identify $Q(A(P, R))$ as the collection of $P \times P$ matrices with entries from R, having at most finitely many nonzero entries in each row and in each column, for which the entry in the x-row, y-column is zero whenever $x \not\leq y$ $(x, y \in P)$. Similarly, we identify $E(A(P, R))$ as the collection of

$P \times P$ matrices with entries from R, having at most finitely many nonzero entries in each row, for which the entry in the x-row, y-column is zero whenever $x \not\leq y$ $(x, y \in P)$.

The setting of posets affords us the chance to give an example of an approximation overring which is not contained in $E(A)$. Specifically, suppose P is infinite but locally finite (i.e., given any $x, y \in P$, there exists only finitely many $z \in P$ such that $x \leq z \leq y$). Let $\hat{A}(P, R)$ denote the collection of $P \times P$ matrices with entries from R (with NO finiteness condition) for which the entry in the x-row, y-column is zero whenever $x \not\leq y$ $(x, y \in P)$. The structure of P ensures that multiplication in $\hat{A}(P, R)$ is well-defined. We leave to the reader the details that $\hat{A}(P, R)$ is a ring of the indicated type. We call such a ring the **completion of** $A(P, R)$. Now to the task at hand.

Theorem 6.2 *Let P be a preordered set such that for every $i \in P$, $\{j \in P : i \leq j$ or $j \leq i\}$ is not finite. Let R be a unital ring and let A denote $A(P, R)$. Then every automorphism of $Q(A)$ restricts to an automorphism of A. As a consequence, $Q(A)$ has the* $\mathrm{Aut}(A)$*-property.*

Proof. Note that $Q = Q(A)$ is the ring of matrices X over R indexed by P satisfying the following conditions:

1. If $X(i, j) \neq 0$ then $i \geq j$.
2. For every $i \in P$, $\{j \in P : X(i, j) \neq 0$ or $X(j, i) \neq 0\}$ is finite.

Now suppose α is an automorphism of Q having $\alpha(A) \not\subseteq A$. Since $A = \sum_{i \in P} Ae_i$ then there exists some $i \in P$, such that $\alpha(e_i) \notin A$. We are going to construct an element s of Q for which $\alpha^{-1}(s) \notin Q$, and hence obtain a contradiction. The hypothesis on P means that either $\{j \in P : i \leq j\}$ or $\{j \in P : j \leq i\}$ is infinite. We will assume that the first set is infinite; a symmetric argument finishes the proof.

We begin by showing that for every $j \geq i$, $\alpha(e_{ji}) \notin A$. Suppose not. Then there exists a finite set F such that $\alpha(e_{ji})e_k = 0$ for every $k \in P \setminus F$. Therefore, there exists $k \in P \setminus F$ such that $\alpha(e_i)e_k \neq 0$ and so $e_i\alpha^{-1}(e_k) \neq 0$. This implies that $e_{ji}\alpha^{-1}(e_k) \neq 0$ which yields a contradiction.

For every $j \geq i$, let

$$X_j = \{x \in P : \alpha(e_{ji})e_x\alpha(e_i) \neq 0\}$$

Using the fact that $\alpha(e_{ji})\alpha(e_i) = \alpha(e_{ji}) \neq 0$, one can show that $X_j \neq \emptyset$. Moreover X_j is not finite. Indeed, assume it is and let $e = \sum_{x\in X_j} e_x$. If $x \geq y$ are elements of P, then $\alpha(e_ji)(x, y) = \sum_{z\in P} \alpha(e_{ji}(x, z)\alpha(e_i)(z, y) = \sum_{z\in X_j} \alpha(e_{ji}(x, z)\alpha(e_i)(z, y) = (\alpha(e_{ji})e\alpha(e_i))(x, y)$. Then $\alpha(e_{ji}) = \alpha(e_{ji})e\alpha(e_i)$ and so this belongs to A and this contradicts the previous paragraph.

Let $(j_n)_{n\in\mathbf{N}}$ be a sequence in $\{j \in P : j \geq i\}$ such that $n \neq m$ implies $j_n \neq j_m$. To simplify the notation set $X_n = X_{j_n}$.

Now we construct by recursion a sequence (k_n) of elements of P and a sequence Y_n of finite subsets of P such that, for every n, $k_n \in X_n \setminus Y_n$. Let k_1 be an element of X_1 and $Y_1 = \emptyset$. Assume we have constructed $k_1, \ldots, k_n$ and $Y_1, \ldots, Y_n$. For every $t = 1, \ldots, n$ let

$$Z_t = \{z \in P : e_z\alpha(e_{j_t i})e_{k_t} \neq 0\}$$

which is finite. Let

$$Y_{n+1} = \{y \in P : (\exists t \leq n, \exists x \in Z_t)e_x\alpha(e_{j_{n+1}i})e_y \neq 0\}$$

which is finite too. Let k_{n+1} be an element of $X_{n+1} \setminus (Y_{n+1} \cup \{k_1, \ldots, k_n\})$.

Let $r = \sum_{n\in\mathbf{N}} \alpha(e_{j_n i})e_{k_n}$. First note that this sum is well defined because the k_n are different. Moreover, by the same reason and the fact that $\alpha(e_{j_n i})e_{k_n} \in Q$, each column of r has only finitely many nonzero entries. We claim that each row of r has only finitely many nonzero entries, and hence $r \in Q$. By the construction it is enough to show that $T = \{n \in \mathbf{N} : r(x, k_n) \neq 0\}$ has at most one element. Assume T is not empty. Let t be the smallest natural number satisfying $k_t \in T$. Then $e_x\alpha(e_{j_t i})e_{k_t} \neq 0$ and hence $x \in Z_t$. Let $n > t$. If $e_x\alpha(e_{j_n i})e_{k_n} \neq 0$ then $k_n \in Y_n$ and hence $e_x\alpha(e_{j_n i})e_{k_n} = 0$. Therefore, $r(x, k_n) = 0$.

We present the element $s \in Q$ which gives the contradiction. Let $s = r\alpha(e_i) \in Q$. Then $\alpha^{-1}(s) \in Qe_i \in A$. However $e_{j_n}\alpha^{-1}(s) \neq 0$ for every n. To see this it suffices to show that $\alpha(e_{j_n})s \neq 0$. But $\alpha(e_{j_n})s = \alpha(e_{j_n i})e_{k_n}\alpha(i) \neq 0$, because $k_n \in X_n$. ▮

There are many examples of rings which satisfy the hypotheses of the above theorem. We mention two below.

Corollary 6.3 *Let R be a ring with identity.*

1. *([8, Proposition 3]) Let $A = FM(R)$ be the ring of countably infinite square matrices with only finitely many nonzero entries from R. Every automorphism of the ring $Q = RCFM(R)$ of row and column finite matrices over R restricts to an automorphism of A.*
2. *Let $A = FLTM(R)$ be the ring of countably infinite square lower triangular matrices with only finitely many nonzero entries from R. Every automorphism of the ring $Q = RCFLTM(R)$ of lower row and column finite triangular matrices over R restricts to an automorphism of A.*

Proof. We leave it to the reader to check that these rings A are incidence rings satisfying the hypotheses of Theorem 6.2 and that the ring Q is the corresponding multiplier ring. ▮

Note that the condition on P in the theorem is necessary because a direct sum of copies of a unital ring is the incidence ring of the preordered set $(\mathbf{N}, =)$ and we've already seen that such rings can fail to satisfy the conclusion of the theorem (see Example 3.2).

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